EVERY THREE-SPHERE OF POSITIVE RICCI CURVATURE CONTAINS A MINIMAL EMBEDDED TORUS

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One of the most celebrated theorems of differential geometry is the 1929 theorem of Lusternik and Schnirelmann, which states that for every riemannian metric on the 2-sphere there exist at least three simple closed geodesics. Jurgen Jost [J] (following important work of Pitts [P] and Simon and Smith [SS]) has recently generalized this result by showing that for every riemannian metric on $S^3$, there exist at least 4 minimal embedded 2-spheres. This is optimal in that there are metrics for which the number of embedded minimal 2-spheres is exactly four [W2,4.5]. However, one can also ask about surfaces of higher genus, and here our knowledge is very incomplete. On the one hand, Lawson [L] showed that $S^3$ with its standard metric contains embedded minimal surfaces of every orientable topological type, and recently Pitts and Rubinstein [PR] have discovered many new infinite families of examples. But for general metrics on $S^3$, no known theorem asserts the existence of any minimal surface other than a sphere. The present paper takes a first step in this direction by proving:

**Theorem 3.** For every $C^4$ riemannian metric $\gamma$ of positive ricci curvature on $S^3$, there exists at least one minimal embedded torus.

I conjecture that every metric on $S^3$ admits at least 5 minimal embedded tori, but I can prove it (by a perturbation argument [W2,4.4]) only for metrics that are close to the standard metric.

**Preliminaries.** The proof uses the following facts about the space of all minimal surfaces for varying riemannian metrics. The facts are proved in [W2] using the implicit function theorem. Let $N$ be a compact 3-manifold, $\Gamma$ be an open set of $C^4$ metrics on $N$, and $M$ be the set of pairs $(\gamma, S)$ where $\gamma \in \Gamma$ and $S \subset N$ is a smooth embedded $\gamma$-minimal surface.

**Theorem 1 [W2, 2.1,2.2,5.1].** The set $M$ is a smooth Banach manifold, and the map

$$\Pi : M \to \Gamma$$

$$\Pi : (\gamma, S) \mapsto \gamma$$

is a smooth map. Almost every (in the sense of Baire category) $\gamma$ is a regular value of $\Pi$, i.e., each element of $\Pi^{-1}(\gamma)$ is a nondegenerate critical point of the area functional.
Let $M_0$ be the union of one or more connected components of $M$. If $\Gamma$ is connected and if $\Pi : M_0 \to \Gamma$ is a proper map (inverse images of compact sets are compact), then $\Pi|_{M_0}$ has a mapping degree $d$ such that for each regular value $\gamma$,

\[(1) \quad d = \sum_{(\gamma, S) \in M_0} (-1)^{\text{index}(S)}.
\]

**Remark 1.** If for some $\gamma$, $\Pi^{-1}(\gamma) = \emptyset$, then $\gamma$ is a regular value of $\Pi$ and so $d$ would have to be $0$ by (1).

**Remark 2.** This theorem remains true for simple immersed minimal surfaces. An immersion is *simple* if there is an $x \in N$ that is covered exactly once.

If $(\gamma, S) \in M_0$ and $S$ has nullity $0$, then $S$ is a nondegenerate critical point of the area functional, and $(\gamma, S)$ is isolated in $\Pi^{-1}(\gamma)$ and contributes $(-1)^{\text{index}(S)}$ to the degree. It sometimes happens that $\Pi^{-1}(\gamma)$ contains a compact $k$-dimensional manifold $\Sigma$ of surfaces. Of course the surfaces in $\Sigma$ are degenerate critical points, but if each of them has nullity equal to $k$, then $\Sigma$ is said to be a nondegenerate critical manifold and has nice properties.

**Theorem 2** [W2,5.1]. Suppose $\Pi^{-1}(\gamma_0)$ contains a nondegenerate critical manifold $\Sigma$. Then there is a neighborhood $\Gamma_0 \subset \Gamma$ of $\gamma_0$ and a connected component $M_0$ of $\Pi^{-1}(\Gamma_0)$ such that

\[\Pi^{-1}(\gamma_0) \cap M_0 = \Sigma\]

and

\[\deg(\Pi|_{M_0}) = \chi(\Sigma)(-1)^{\text{index}(\Sigma)}\]

where $\chi(\Sigma)$ is the euler characteristic of $\Sigma$ and $\text{index}(\Sigma)$ is equal to the index of $S$ for each $(\gamma, S) \in \Sigma$.

For example consider the clifford tori. (A clifford torus is the set of points in $S^3 \subset R^4$ equidistant from a pair of orthogonal planes through the origin.) The set of all such tori (or, equivalently, the set of pairs of orthogonal planes in $R^4$) is topologically the 4-manifold $RP^2 \times RP^2$. Straightforward calculations show that each clifford torus is a minimal surface with nullity $4$ and index $1$. Thus the hypotheses of Theorem 2 are satisfied.

**Theorem 3.** For every $C^4$ metric $\gamma$ of positive ricci curvature on $S^3$, there exists at least one embedded torus that is minimal with respect to $\gamma$.

**Proof.** Let $\gamma_0$ be the standard metric on $S^3$ and let $S^1 = \{z \in C : |z| = 1\}$ act on $S^3 \subset R^4 = C^2$ by complex multiplication. Let $R_n : S^3 \to S^3$ be multiplication by $e^{2\pi i/n}$. We first prove the following:

**Lemma.** There is an $n_0$ such that if $n \geq n_0$ and if $M \subset S^3$ is an embedded $\gamma_0$-minimal torus with $R_n(M) = M$, then $M$ is a clifford torus.
PROOF. Suppose not. Then there is a sequence $n(i) \to \infty$ of integers and a sequence $M_i$ of embedded minimal tori, none of which is a Clifford torus, such that $R_{n(i)}(M_i) = M_i$.

By the compactness theorem of Choi and Schoen [CS], there is a convergent subsequence, which we may assume to be the original sequence: $M_i \to M$. Fix a real number $t > 0$ and let $k(i)$ be the greatest integer less than or equal to $tn(i)$, so that

$$\frac{k(i)}{n(i)} \to t.$$ 

By hypothesis

$$e^{\frac{2\pi}{n(i)}(M_i)} = M_i.$$ 

Thus

$$e^{2\pi i \frac{k(i)}{n(i)}(M_i)} = \left(e^{\frac{2\pi i}{n(i)}}\right)^{k(i)}(M_i) = M_i$$

so, letting $i \to \infty$,

$$e^{2\pi it}(M) = M.$$ 

Since this holds for each $t$, $M$ is invariant under the $S^1$ action. Let

$$h : S^3 \subset C^2 \to S^2 \cong C \cup \{\infty\}$$

be the Hopf map

$$h(z, w) = z/w.$$ 

In more geometrical terms, $h : S^3 \subset C^2 \to CP^1 \cong S^2$ maps each point to the complex line that contains it. The $S^1$ invariance of $M$ means that there is a curve $T \subset S^2$ such that

$$M = h^{-1}(T).$$

The area of the inverse image (under $h$) of a curve is a constant times the length of the curve. Thus the minimality of $M$ implies that $T$ is a union of geodesics. Since $M$ is embedded, $T$ is a single great circle and so $M$ is a Clifford torus. It follows from Theorem 2 (and the example given after it) that for sufficiently large $i$, $M_i$ is a Clifford torus. □

Now to prove the theorem, let $\Gamma$ be the set of $C^4$ metrics of positive Ricci curvature on $S^3$, $\mathcal{M}$ be the Banach manifold of Theorem 1, and $\mathcal{M}_0$ be the set of $(\gamma, S) \in \mathcal{M}$ such that $S$ is a torus. The space $\Gamma$ is connected by a theorem of Hamilton [H]. By the compactness theorem of Choi and Schoen [CS], $\Pi|\mathcal{M}_0$ is proper and therefore (by Theorem 1) has a mapping degree; we will show that the degree is not zero. Let $p$ be a prime number greater than the $n_0$ of the lemma. Let $\gamma_0$ be the standard metric on $S^3$ and let $\Gamma_0 \subset \Gamma$ be a neighborhood of $\gamma_0$ such that $\Pi^{-1}(\Gamma_0)$ contains a connected component $\mathcal{M}_{\text{cliff}}$ such that

$$\Pi^{-1}(\gamma_0) \cap \mathcal{M}_{\text{cliff}}$$

is precisely the set of Clifford tori. (This is possible by Theorem 2.) Let

$$\mathcal{M}_{\text{other}} = \Pi^{-1}(\Gamma_0) \cap \mathcal{M}_0 \setminus \mathcal{M}_{\text{cliff}}.$$
By the lemma we may choose $T_0$ small enough that if $(\gamma, S) \in M_{\text{other}}$ then $R_p(S) \neq S$. Note that since $p$ is prime, if $T \subset S^3$ is an embedded submanifold that is not $R_p$ invariant, then there is an $x \in T$ such that $x \notin (R_p)_k(T)$ for $1 \leq k < p$. That is, $T$ becomes a simply immersed submanifold in the quotient space $S^3/R_p$. According to Theorem 1, for almost every metric $\gamma$ on $S^3/R_p$, each simple $\gamma$-minimal surface has nullity 0. Equivalently, for almost every $R_p$ invariant metric $\gamma$ on $S^3$, each $\gamma$-minimal surface that is not $R_p$ invariant has nullity 0. Fix any such $R_p$ invariant metric $\gamma \in \Gamma_0$.

Now if $(\gamma, S) \in M_{\text{other}}$, then so is $(\gamma, (R_p)_kS)$ for $1 \leq k < p$. Furthermore, these $p$ surfaces are distinct and all have the same index (and nullity 0). Thus

$$\deg(\Pi|_{M_{\text{other}}}) = \sum_{(\gamma, S) \in M_{\text{other}}} (-1)^{\text{index}(S)} \equiv 0 \mod p.$$ 

On the other hand, by Theorem 2 and the example following it:

$$\deg(\Pi|_{M_{\text{cliff}}}) = \chi(RP^2 \times RP^2)(-1)^1 = -1.$$ 

Thus

$$\deg(\Pi|_{M_0}) = \deg(\Pi|_{M_{\text{cliff}}}) + \deg(\Pi|_{M_{\text{other}}}) \equiv -1 \mod p.$$ 

Since this holds for arbitrarily large $p$, in fact $\deg(\Pi|_{M_0})$ must be $-1$. The theorem follows immediately (see Remark 1 after Theorem 1). 

\textbf{Remark 1.} The argument above for $M_{\text{other}}$ also shows

\textbf{Theorem 4.} Let $M_g$ be the Banach manifold of pairs $(\gamma, S)$ where $\gamma$ is a $C^4$ metric of positive ricci curvature on $S^3$ and $S \subset S^3$ is an embedded surface of genus $g + 1$ that is minimal with respect to $\gamma$. Then

$$\deg(\Pi|_{M_g}) = 0$$ 

unless $g = 0$.

\textbf{Remark 2.} Mapping degrees were first applied to minimal surface theory by Tomi and Tromba [TT]. Additional applications were given in [W1]. Theorem 3 is the first result for which it is necessary to use the integral mapping degree rather than the simpler mod 2 degree.

\textbf{References}


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