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MELVIN HENRIKSEN
HARVEY MUDD COLLEGE

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

0273-0979/89 $1.00 + $.25 per page

Introduction to Arakelov theory, by Serge Lang. Springer-Verlag, New York, Berlin, Heidelberg, 1988, x + 187 pp., $49.95. ISBN 0-387-96793-1

In the beginning (or shortly thereafter) there were complex projective surfaces. These algebraic subsets of $\mathbf{P}^n(\mathbf{C})$ of complex dimension 2 were extensively studied by Italian geometers, such as Castelnuovo, Enriques, and Severi, during the late nineteenth and early twentieth century. It soon became apparent that the key to understanding such surfaces is to study the curves which they contain. Thus if $X \subset \mathbf{P}^n(\mathbf{C})$ is a smooth surface, one looks at the curves $C \subset X$; and, more generally, one looks at the free abelian group generated by these curves, which is called the *group of divisors on* X and is denoted

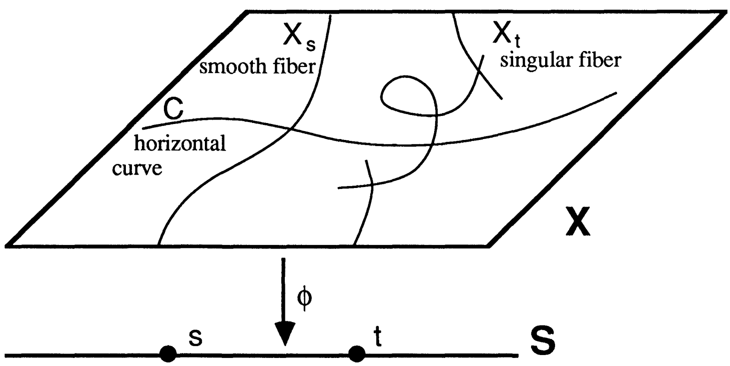
$$\operatorname{Div}(X) = \left\{ \sum_{C \subset X} n_C[C] \colon n_C \in \mathbf{Z},\ \text{almost all } n_C = 0 \right\}.$$

Associated to a rational function f on X is its set of zeros and poles; taken with multiplicities, these zeros and poles give a divisor. Two divisors $D_1, D_2 \in \operatorname{Div}(X)$ are called *linearly equivalent* if their difference $D_1 - D_2$ is the divisor of a function.

Given two distinct curves C_1 and C_2 on X, one can count the number of points where they intersect (with multiplicity, if the intersection is not transversal). Extending this intersection index linearly to $\operatorname{Div}(X)$ gives the *intersection pairing*

$$\langle\ ,\ \rangle \colon \operatorname{Div}(X) \times \operatorname{Div}(X) \to \mathbf{Z},$$

defined a priori for divisors with no common components. An important property of the intersection pairing is that it is invariant under linear equivalence. (I.e. If D_1 is linearly equivalent to D_2, then $\langle D, D_1\rangle = \langle D, D_2\rangle$ for all $D \in \mathrm{Div}(X)$.) This allows one to move a divisor, and so to define $\langle D_1, D_2\rangle$ for all pairs of divisors. Virtually all of the major theorems in the classical theory, such as the Riemann-Roch theorem, Hodge index theorem, Castelnuovo's criterion, and Noether's formula, involve divisors and their intersections. (See, for example, [**9**, Chapter 4 or **10**, Chapter 5].)



A Fibered Surface

FIGURE 1

One way to study the curves on a smooth surface is by looking at the fibers of a map $\phi : X \to \mathbf{P}^1$; or more generally, of a map $\phi : X \to S$ to an arbitrary curve S. For all but finitely many points $s \in S$, the fiber $X_s = \phi^{-1}(s)$ will be a single smooth curve; and there will be some finite set of points $\{s_1, \ldots, s_r\}$ whose fibers X_{s_i} consist of one or more possibly singular curves. This fibration divides the curves on X into two sorts, those which are fibral and those for which the map $\phi : C \to S$ is a finite covering. (See Figure 1.)

During the 1960s Grothendieck suggested studying fibrations $\phi : X \to S$, where S is no longer an algebraic variety. For example, let $S = \mathrm{Spec}(\mathbf{Z})$, the set of prime ideals in $\mathbf{Z}$. Then X is given by the zeros of a collection of polynomial equations with integer coefficients; and for a point $s = (p) \in S$, the "fiber" X_s consists of the same equations reduced modulo p. For example, suppose that X is the subset of $\mathbf{P}^2$ given by the single homogeneous equation

$$X : x^3 + y^2 z + z^3 = 0.$$

The complex solutions to this equation, denoted $X_\infty(\mathbf{C})$, give a smooth, projective curve of genus 1 (a so-called "elliptic curve"). The fiber of $\phi : X \to S$ lying over the point $s = (p)$ is a curve over the finite field $\mathbf{F}_p$:

$$X_s = \{[x, y, z] \in \mathbf{P}^2(\overline{\mathbf{F}}_p) : x^3 + y^2 z + z^3 = 0\}.$$

Then X_s is a singular curve if $p = 2$ or 3, and a smooth curve for all other primes. If $X_\infty(\mathbf{C})$ is a curve, as in this example, then one says that X is an *arithmetic surface.*[1]

Since an arithmetic surface X is supposed to be an analogue of the geometric surfaces described above, one can ask to study the "curves" on X and to develop an intersection theory. There are again two sorts of curves on X. First, there are the fibers X_s (and their components, if X_s is reducible). Second, there are the horizontal curves $C \subset X$, those for which the map $\phi : C \to S$ is a finite covering. If this map has degree d, then the curve C corresponds to a point on X whose coefficients lie in a field of degree d over $\mathbf{Q}$. Thus classical Diophantine questions about rational points on curves (e.g. Mordell's conjecture) have a natural interpretation in terms of curves on arithmetic surfaces.

Next one tries to calculate the intersection of curves on an arithmetic surface. For distinct horizontal curves, the classical definition using local intersection indices essentially works. Lichtenbaum and Shafarevich have shown that some classical theorems, such as Castelnuovo's criterion on contractibility of exceptional divisors and certain embedding theorems, can then be generalized. (See **[6, 13, 14]**.)

However, most of the classical theory depends on the fact that the intersection pairing is invariant under linear equivalence. And the underlying reason that this is true is that the surface X is complete (i.e. compact). Unfortunately, an arithmetic surface $X \to S$ is not "compact." The base curve $S = \mathrm{Spec}(\mathbf{Z})$ is the analogue of the affine line $\mathbf{A}^1$, not the projective line $\mathbf{P}^1$. So X is not complete because it is missing a fiber "at infinity." Arakelov observed that since the points of $\mathrm{Spec}(\mathbf{Z})$ correspond to the p-adic absolute values on $\mathbf{Z}$, the missing point of $\mathrm{Spec}(\mathbf{Z})$ should correspond to the usual absolute value (i.e. the one induced by $\mathbf{Z} \subset \mathbf{R}$). Call this extra point ∞, and write $S^* = \mathrm{Spec}^*(\mathbf{Z})$ for $\mathrm{Spec}(\mathbf{Z}) \cup \{\infty\}$. Then the fiber X_∞ is just the complex curve $X_\infty(\mathbf{C})$; and with this extra fiber, the arithmetic surface $X^* = X \cup X_\infty$ is complete.

This idea of considering all absolute values on $\mathbf{Z}$ is a number theoretic technique which dates back to the late nineteenth century, and it has been extensively used ever since. Arakelov's great insight was to suggest how to "complete" the intersection theory at infinity. In brief, his idea is as follows. The problem is to calculate to what extent two horizontal curves $C_1, C_2 \subset X$ intersect on the fiber at infinity. These curves correspond to points P_1, P_2 in $X_\infty(\mathbf{C})$, so intuitively one wants

$$\langle C_1, C_2 \rangle_\infty = -\log(\text{distance from } P_1 \text{ to } P_2 \text{ on } X_\infty(\mathbf{C})).$$

(Note that for finite primes p, the intersection $\langle C_1, C_2 \rangle_p$ corresponds to the p-adic distance.) Arakelov takes a certain normalized, logarithmic

[1] Note for experts: Many of the definitions and theorems quoted in this review are only approximately correct. For example, an arithmetic surface (as defined in Lang) is integral, proper over its Dedekind domain base, and has smooth generic fiber. I have sacrificed the rigor of such definitions in favor of brevity and clarity suitable to a review. I have also generally ignored the important issues of regularity and semistability, which Lang (rightly) treats in some detail.

Green's function $g(P,Q)$ on the Riemann surface $X(\mathbf{C})$. This is a real analytic function on $X(\mathbf{C}) \times X(\mathbf{C})$ with a logarithmic singularity along the diagonal. He then defines

$$\langle C_1, C_2 \rangle_\infty = g(P_1, P_2).$$

With this extra contribution to the intersection index, Arakelov shows that the intersection pairing is invariant under linear equivalence. He then proves (under certain restrictions) an arithmetic adjunction formula, which relates the self-intersection $\langle D, D\rangle$ to the intersection $\langle D, K_X\rangle$ of D with an arithmetic canonical divisor K_X. (See **[2, 3, 5]**.)

Two of the most fundamental theorems in the classical theory of algebraic surfaces are the Riemann-Roch Theorem and the Hodge Index Theorem. The Riemann-Roch Theorem expresses the dimension of certain cohomology groups in terms of intersection theory. Precisely, define the Euler characteristic of a divisor by

$$\chi(D) =: \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) + \dim H^2(X, \mathcal{O}_X(D)).$$

Then the Riemann-Roch Theorem says

$$\chi(D) - \chi(0) = \tfrac{1}{2}\langle D, D - K_X\rangle. \tag{1}$$

Thus the Riemann-Roch theorem expresses the size of certain cohomology groups in terms of an intersection index. Faltings' arithmetic version of Riemann-Roch is similar. He defines a volume form on the vector spaces $H^i(X, \mathcal{O}_X(D)) \otimes \mathbf{C}$ (really on a certain alternating tensor product). Then the volume of a fundamental domain for the lattice $H^i(X, \mathcal{O}_X(D))$ inside $H^i(X, \mathcal{O}_X(D)) \otimes \mathbf{C}$ measures how large $H^i(X, \mathcal{O}_X(D))$ is. Faltings shows that an Euler characteristic defined with these volumes can be expressed in terms of Arakelov intersections. In fact, his formula looks exactly like (1). Faltings (and Hriljac independently) also gave proofs of an arithmetic analogue of the Hodge Index Theorem.

After this lengthy prologue, we come to the book under review. In a brief 154 pages of text, Lang has covered most of the material discussed above and more. He begins (Chapter I) with a discussion of "line sheaves" (his term for invertible sheaves) and metrics on them. This chapter will be easiest for those who have read his discussion of Weil functions and Néron divisors in [**12**, Chapter 10]. He next (Chapter II) gives the (technical) construction of the Green's functions needed for the definition of Arakelov's intersection theory. This is purely analytic, and the reader willing to accept the existence of these functions can just read the statements of the theorems in this chapter. The author continues (Chapter III) with an exposition of intersection theory on an arithmetic surface $X \to \operatorname{Spec}(R)$, as developed by Lichtenbaum [**13**], Shafarevich [**14**], and Néron [**12**, Chapter 11, §3]. This is the pre-Arakelov intersection theory which lacks the fiber(s) at infinity.

The infinite fiber(s) are added next (Chapter V), and the invariance of the Arakelov intersection pairing under linear equivalence is proven. The arithmetic Hodge Index Theorem is proven as a consequence of the positive definiteness of the Néron-Tate height, for which the reader is referred to [12]. The canonical bundle is constructed and metrized, and the residue theorem is proven (with reference to [11] for some details). The chapter concludes with a proof of Arakelov's arithmetic adjunction formula. This proof is notable for the fact that it works directly on the arithmetic surface X; it neither requires semistability, nor does it make a base extension to reduce to this case. The last two chapters (Chapters V and VI) contain the proof of Faltings' arithmetic Riemann-Roch Theorem, again given without the use of semistability assumptions, and some applications of Faltings' theorem. The hardest part of the proof is showing that it is possible to assign volume forms (Faltings metric) to the determinants of certain cohomology groups in a consistent fashion. This result is stated in Chapter V, and the proof occupies most of Chapter VI. Granting the existence of the Faltings metric, the proof of the Riemann-Roch Theorem is similar to the classical case; one works inductively, adding one irreducible component at a time to the divisor. Chapter VI contains generalities on determinants of derived sheaves, a proof of the existence of the Faltings metric (modulo some standard facts on the Θ-divisor that can be found in [4]), and a result of Elkies bounding certain averages of a Green's function.

In a short appendix, Paul Vojta describes Parshin's (conjectural) arithmetic analogue of the famous $c_1^2 \leq 3c_2$ inequality for complex algebraic surfaces, relates Parshin's question to some of his own conjectures [15], and gives various Diophantine applications.

This briefly describes what is in Lang's book. Some related topics that were not included are: Faltings' arithmetic analogue of Noether's formula, recent progress on higher dimensional arithmetic intersection theory due to Gillet and Soulé, Deligne, Quillen, and others; applications of arithmetic intersection theory to physics; Vojta's recent independent proof of Mordell's conjecture (Faltings' theorem) using (among other tools) the Riemann-Roch Theorem on arithmetic three-folds. These advanced topics would make a nice companion volume.

Many of the proofs in Lang's book are really closer to proof sketches, either leaving details for the reader to check, or referring to other sources for major pieces of the proof. Lang makes frequent use of material proven in Hartshorne [**10**, especially Chapters II and III], Griffiths-Harris [**9**, Chapter 0], Altman-Kleinman [**1**], and his own book on Diophantine geometry [**12**]. Other sources are occasionally cited. This is clearly a compromise needed to enable him to cover so large an amount of material in so few pages. But it will put a strain on the reader not having the "rather vast background required for its reading."[2] It would certainly be useful to have a book of 400 or 500 pages on the same material, containing many more details, examples, and exercises. Such a book would "coexist amicably" with Lang's, and "neither would be better than the other." (One would be better for students, the other for mathematicians already having the necessary background.) But until someone makes the major effort necessary

[2]This quote and those following are taken (slightly out of context) from a letter written by Lang to Mordell concerning the first edition of [12] and published in an appendix to the second edition.

to write such a text, Lang's book is sure to be the standard reference for the basic material in this important and active area of research.

Finally, the obligatory comments on errors, typos, etc. The typesetting is nicely done, and I found only a few minor glitches (e.g. "variables" in place of "variable" on p. 11, "models" in place of "modules" on p. 106). One should also note a possible source of confusion concerning the important theorem giving the existence of the Faltings metric. This is stated as Theorem 3.2 of Chapter V, where the Faltings metric is described as having four properties **FAL1**, ..., **FAL4**. The proof is deferred to Chapter VI, where the theorem is restated as Theorem 3.1. But now there are only three properties **FAL1**, ..., **FAL3**! One easily checks that Lang has combined the old properties **FAL2** and **FAL4** into the new **FAL2**.

In conclusion, Lang has written a valuable introduction to the basic theory in this new and important area of arithmetic geometry. Much of the material he covers was previously available only in the original journal articles, and he is to be commended for bringing it together in such a coherent fashion. It will undoubtedly become a standard reference, and certainly belongs on the bookshelf of any serious mathematician working in arithmetic algebraic geometry or the theory of Diophantine equations.

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JOSEPH H. SILVERMAN
BROWN UNIVERSITY