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FINITENESS AND VANISHING THEOREMS FOR COMPLETE OPEN RIEMANNIAN MANIFOLDS

ZHONGMIN SHEN

Let M^n denote an n-dimensional complete open Riemannian manifold. In [AG] Abresch and Gromoll introduced a new concept of "diameter growth." Roughly speaking, one would like to measure the essential diameter of ends at distance r from a fixed point $p \in M^n$. They showed that M^n is homotopy equivalent to the interior of a compact manifold with boundary if M^n has nonnegative Ricci curvature and diameter growth of order $o(r^{1/n})$, provided the sectional curvature is bounded from below. It is well known that any complete open manifold with nonnegative sectional curvature has finite topological type. This is a weak version of the *Soul Theorem of Cheeger-Gromoll* [CG]. Examples of Sha and Yang show that this kind of finiteness result does not hold for complete open manifolds with nonnegative Ricci curvature in general (see [**SY1**, **SY2**]), and additional assumptions are therefore required.

We will use a concept of the essential diameter of ends slightly stronger than that of [AG]: For any $r > 0$, let $B(p,r)$ denote the geodesic ball of radius r around p. Let $C(p,r)$ denote the union of all unbounded connected components of $M^n \backslash \overline{B(p,r)}$. For $r_2 > r_1 > 0$, set $C(p;r_1,r_2) = C(p,r_1) \cap B(p,r_2)$. Let $1 > \alpha > \beta > 0$ be fixed numbers. For any connected component Σ of $C(p;\alpha r, \frac{1}{\alpha} r)$, and any two points $x, y \in \Sigma \cap \partial B(p,r)$, consider the distance $d_r(x,y) = \inf \mathrm{Length}(\phi)$ between x and y in $C(p,\beta r)$, where the infimum is taken over all smooth curves $\phi \subset C(p,\beta r)$ from x to y. Set $\mathrm{diam}(\Sigma \cap \partial B(p,r), C(p,\beta r)) = \sup d_r(x,y)$, where $x, y \in \Sigma \cap \partial B(p,r)$. Then the diameter of ends at distance r from p is defined by

$$\mathrm{diam}(p,r) = \sup \mathrm{diam}\left(\Sigma \cap \partial B(p,r), C(p,\beta r)\right),$$

where the supremum is taken over all connected components Σ of $C(p;\alpha r, \frac{1}{\alpha} r)$. The *diameter* defined here is not smaller than that defined by Abresch and Gromoll. Our definition will be essential in Lemma 3 and its applications.

The purpose of this note is to announce the following results.

THEOREM A. *Let M be a complete open Riemannian manifold with sectional curvature $K_M \geq -K^2$ for some constant $K > 0$. Assume that for some base point $p \in M$,*

$$\limsup_{r \to +\infty} \mathrm{diam}(p,r) < \frac{\ln 2}{K}.$$

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Then M is homotopy equivalent to the interior of a compact manifold with boundary.

THEOREM B. *Let M^n be an n-dimensional complete open Riemannian manifold. Suppose that the sectional curvature $K_M \geq -K^2$ for some constant $K > 0$. Assume that for some $2 \leq k \leq n-1$, M^n has nonnegative kth-Ricci curvature and that for some $p \in M^n$,*

$$\limsup_{r\to+\infty} \frac{\operatorname{diam}(p,r)}{r^{\frac{1}{k+1}}} < \left[\frac{2(k+1)}{k}\left(\frac{(k-1)\ln 2}{2kK}\right)^k\right]^{1/(k+1)}.$$

Then M^n is homotopy equivalent to the interior of a compact manifold with boundary.

THEOREM C. *Let M^n be an n-dimensional complete open Riemannian manifold. Assume that for some $1 \leq k \leq n-1$, M^n has positive kth-Ricci curvature everywhere and that for some $p \in M^n$, M^n has diameter growth of order $o(r)$, i.e.*

$$\limsup_{r\to+\infty} \frac{\operatorname{diam}(p,r)}{r} = 0.$$

Then M^n has the homotopy type of a CW-complex with cells of dimensions $\leq k-1$.

The precise condition that M^n have nonnegative (positive) kth-Ricci curvature at some point $x \in M^n$ is that for all v in the span of any orthonormal set $\{e_1, \dots, e_{k+1}\}$ in T_xM^n,

$$\sum_{i=1}^{k+1} \langle R(e_i, v)v, e_i\rangle \geq 0 \ (> 0),$$

where $R(x,y)z$ denotes th curvature tensor of M^n (cf. also **[H]** for the definition of kth-Ricci curvature).

REMARK 1. (1) In Theorem A the upper bound $\ln 2/K$ must depend on K. Otherwise, the connected sum of infinitely many copies of $S^2 \times S^2$ (see **[AG]**) provides an easy counterexample.

(2) Theorem B generalizes the *Abresch-Gromoll Theorem* **[AG]**.

(3) The condition in Theorem C can be weakened to that M^n has nonnegative kth-Ricci curvature everywhere and positive kth-Ricci curvature outside a compact subset of M^n (see Lemma 5).

(4) The same argument as in **[AG]** shows that any complete open Riemannian manifold with nonnegative Ricci curvature must have diameter growth of order $o(r)$. We do not know whether the condition in Theorem C on diameter growth is necessary. Examples in **[SY1, SY2, We** and **GM]** have diameter growth of order at most $o(r)$.

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OUTLINE OF PROOFS. Throughout this part we assume that M^n denotes a complete open Riemannian manifold of dimension n and p is a point of

M^n fixed during the discussion. For arbitrary $t \geq 0$, let $R_t(p) = \{\gamma(t); \gamma$ is a ray emanating from $p\}$, which is a closed subset of the distance sphere $S(p,t)$. Set $B_p^t(x) = t - d(x, R_t(p))$ for any $x \in M^n$. It is easy to see that $B_p^t(x)$ is increasing in t and $|B_p^t(x)| \leq d(p,x)$ for any $x \in M^n$. The generalized *Busemann function* B_p is defined as $B_p(x) = \lim_{t\to+\infty} B_p^t(x)$, which is a Lipschitz function with Lipschitz constant 1. The *excess function* E_p is defined as $E_p(x) = d(p,x) - B_p(x)$. We will introduce a new function L_p which plays an essential role in the study of the generalized Busemann function B_p. Set $L_p(x) = d(x, R_t(p))$, where $t = d(p,x)$. Since $B_p^t(x)$ is increasing in t, it is easy to see that $E_p(x) \leq L_p(x)$ and $d(p,x) - L_p(x) \leq B_p(x)$ for all $x \in M^n$. A more detailed discussion for generalized Busemann functions has been given by H. Wu **[W1]**. For the purpose of this note, we need the following

LEMMA 1. *For any $q \in M^n$, there exists a ray $\sigma_q(t)$ emanating from q such that for all $t \geq 0$, the function $B_p^{q,t}(x)$ defined by $B_p(q) + t - d(x, \sigma_q(t))$ supports $B_p(x)$ at q, namely $B_p^{q,t}(x) \leq B_p(x)$ for all $x \in M^n$ and $B_p^{q,t}(q) = B_p(q)$.*

LEMMA 2. *Suppose that M^n has sectional curvature $K_M \geq -K^2$ for some $K > 0$, then for any critical point q with respect to p,*

$$E_p(q) \geq \frac{1}{K}\left(\frac{e^{Kd(p,q)}}{\cosh Kd(p,q)}\right).$$

Notice that $E_p(x) \leq L_p(x)$ for all $x \in M^n$. Thus if $\limsup_{d(p,x)\to+\infty} L_p(x) < \frac{\ln 2}{K}$, Lemma 2 shows that outside a compact subset there is no critical point with respect to p, Theorem A follows from this argument and the following

LEMMA 3. *Suppose that M^n has diameter growth of order $o(r)$. Then there exists an $R > 0$ such that for any $x \in M^n \backslash B(p,R)$,*

$$L_p(x) \leq \operatorname{diam}(p, d(p,x)), \tag{1}$$

and the Busemann function B_p is proper.

Notice that $d(p,x) - L_p(x) \leq B_p(x)$ for all $x \in M^n$. It is clear that (1) implies that $g(x) \equiv d(p,x) - L_p(x)$ is proper, and so is $B_p(x)$.

One can obtain a better estimate for $E_p \leq L_p$ in terms of L_p if M^n has nonnegative kth-Ricci curvature.

LEMMA 4. *Suppose that M^n has nonnegative kth-Ricci curvature for some $2 \leq k \leq n-1$, then for all $x \in M^n$ with $L_p(x) < d(p,x)$,*

$$E_p(x) \leq \frac{2k}{k-1}\left[\frac{k}{2(k+1)} \times \frac{L_p(x)^{k+1}}{d(p,x) - L_p(x)}\right]^{1/k}. \tag{2}$$

The proof of Lemma 4 depends on Lemma 1 and the *maximum principle*. Theorem B therefore follows from Lemmas 2, 3, and 4. For the proof of Theorem C, we need Lemma 3 and the following

LEMMA 5. *Suppose that for some $1 \leq k \leq n-1$, M^n has nonnegative kth-Ricci curvature everywhere and positive kth-Ricci curvature outside a compact subset. If the Busemann function B_p is proper, then there exists a C^2 function $\chi(t)$ such that $\chi \circ B_p$ is proper and strictly k-convex. Therefore M^n has the homotopy type of a CW-complex with cells of dimensions $\leq k-1$.*

Compare **[W2]** for a definition of *k-convexity*. It seems to be crucial that the Busemann function B_p is proper. The first assertion in Lemma 5 follows from Lemma 1. If we assume that $\chi \circ B_p$ is proper and strictly k-convex, then the last assertion in Lemma 5 follows from Wu's Smoothing Theorem **[W2]** and the standard Morse Theory **[M]**. This proves Theorem C.

REMARK 2. An observation of Cheeger-Gromoll (**[CG]**, sharpened in **[GW]**) is that if M^n has nonnegative sectional curvature outside a compact subset, then M^n has finite topological type and B_p is a proper function. If an addition, M^n has nonnegative kth-Ricci curvature everywhere and positive kth-Ricci curvature outside a compact subset, then M^n has the homotopy type of a CW-complex with finitely many cells of dimensions $\leq k-1$ (cf. **[W2]**).

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DEPARTMENT OF MATHEMATICS, SUNY AT STONY BROOK, STONY BROOK, NEW YORK, 11794