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A SURPRISING HIGHER INTEGRABILITY PROPERTY OF MAPPINGS WITH POSITIVE DETERMINANT

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Introduction. Let Ω be a bounded, open set in $\mathbf{R}^n$, $n \geq 2$, and assume that $u\colon \Omega \to \mathbf{R}^n$ belongs to the Sobolev space $W^{1,n}(\Omega;\mathbf{R}^n)$, i.e. $\|u\|^n_{W^{1,n}} = \int_\Omega |u|^n + |Du|^n\,dx < \infty$, where Du denotes the distributional derivative. Then $\det Du$ is, of course, integrable. The aim of this note is to show that under the additional assumption that $\det Du \geq 0$ (almost everywhere) in fact $\det Du \ln(2 + \det Du)$ is integrable (on compact subsets K of Ω). When applied to a sequence of mappings $u^j\colon \Omega \to \mathbf{R}^n$ with $\det Du \geq 0$, $\|u^{(j)}\|_{W^{1,n}} \leq C$, this higher integrability result implies that the sequence $\det Du^{(j)}$ is weakly relatively compact in $L^1(K)$. This allows us to improve known results on weak continuity of determinants **[R, B]** and existence of minimizers in nonlinear elasticity **[BM]**. In the terminology of Lions **[L1, L2]** and DiPerna and Majda **[DM]**, the constraint $\det Du^{(j)} \geq 0$ prevents the development of 'concentrations' in the sequence $\det Du^{(j)}$.

One might ask whether analogous results hold for orientation preserving mappings between oriented compact Riemannian manifolds. In short, the function $\det Du\ \ln(2 + \det Du)$ is still integrable, but not necessarily uniformly so along a sequence which is bounded in $W^{1,n}$. 'Concentrations' may occur, but only in a particular fashion (see **[M]**).

THEOREM 1. *Let $\Omega \subset \mathbf{R}^n$ be bounded and open and let $u\colon \Omega \to \mathbf{R}^n$ be in $W^{1,n}(\Omega;\mathbf{R}^n)$, $n \geq 2$. Assume that* $\det Du \geq 0$ *a.e. Then, for every compact set $K \subset \Omega$,* $\det Du \ln(2 + \det Du) \in L^1(K)$ *and*

$$\|\det Du \ln(2 + \det Du)\|_{L^1(K)} \leq C(K, \|u\|_{W^{1,n}(\Omega)}). \tag{1}$$

The result is optimal in the following sense. The assumption $\det Du \geq 0$ cannot be dropped nor can K be replaced by Ω (see Ball-Murat **[BM**, Counterexample 7.3]). Moreover $\det Du \ln(2+\det Du)$ cannot be replaced by $\gamma(\det Du)$ with $\gamma(z)/(z\ln(2 + z)) \to +\infty$ for $z \to +\infty$ (see **[M]**).

Two key lemmas. The proof of Theorem 1 relies on a geometric estimate (a version of the isoperimetric inequality) and an analytic result on maximal functions by Stein [S2]. We begin with the former. For an $n \times n$ matrix F let $\operatorname{adj} F$ denote the transpose of the matrix of cofactors, so that $F \operatorname{adj} F = \det F\ \mathrm{Id}$.

LEMMA 2. *Let $\Omega \subset \mathbf{R}^n$ be bounded and open and let $u \in W^{1,n}(\Omega;\mathbf{R}^n)$. For $x \in \Omega$ let $B_d(x)$ be a ball of radius d around x such that $B_d(x) \subset \Omega$.*

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Then, for a.e. $r \in (0, d)$,

$$\left| \int_{B_r(x)} \det Du \, dy \right|^{(n-1)/n} \leq c \int_{\partial B_r(x)} |\operatorname{adj} Du| dS, \tag{2}$$

where the constant c *depends only on* n.

If u is a C^1-diffeomorphism, (2) follows from the usual isoperimetric inequality as the left-hand side is $\{\operatorname{vol} u(B_r)\}^{(n-1)/n}$ while the right-hand side is an upper bound for area $u(\partial B_r)$ times a constant. As stated, Lemma 2 is an immediate consequence of the isoperimetric inequality for currents (see Federer [**F**, Theorem 4.5.9 (31)]); an elementary proof, based on approximation by smooth functions and degree theory is also available.

Recall that for $f \in L^1(\mathbf{R}^n)$ the maximal function Mf is defined by

$$Mf(x) = \sup_{R>0} \frac{1}{\operatorname{meas} B_R(x)} \int_{B_R(x)} |f(y)| dy.$$

LEMMA 3 (STEIN [**S2**]). *Let* $f \in L^1(\mathbf{R}^n)$ *and assume that* f *is supported on a ball* B *and that* $Mf \in L^1(B)$. *Then* $|f| \ln(2 + |f|) \in L^1(B)$ *and*

$$\| |f| \ln(2 + |f|) \|_{L^1(B)} \leq C(B, \|Mf\|_{L^1(B)}). \tag{3}$$

Estimate (3) is implicit in [**S1**, p. 23,**S2**], though not explicitly stated.

PROOF OF THEOREM 1. Fix $K \subset \Omega$, compact and let

$$g = 1_K \det Du,$$

1_K being the characteristic function of K. By Lemma 3 we only have to show that the maximal function Mg satisfies

$$\|Mg\|_{L^1(B)} \leq C(K, \|u\|_{W^{1,n}(\Omega)}), \tag{4}$$

for some ball $B \supset \Omega$. Let $d = \operatorname{dist}(K, \partial\Omega)$. It suffices to estimate

$$\frac{1}{\operatorname{meas} B_R(x)} \int_{B_R(x)} |g(y)| dy, \tag{5}$$

for x satisfying $\operatorname{dist}(x, \partial\Omega) > d/2$ and for $R < d/4$, as otherwise (5) is bounded by $C(d)\|u\|_{W^{1,n}(\Omega)}$.

Using the fact that $\det Du \geq 0$ and Lemma 2 we have, for a.e. $r \in (R, 2R)$,

$$\left\{ \int_{B_R(x)} |g(y)| dy \right\}^{(n-1)/n} \leq \left\{ \int_{B_r(x)} \det Du \, dy \right\}^{(n-1)/n} \leq c \int_{\partial B_r(x)} |\operatorname{adj} Du| dS.$$

Here and in the following we denote by c any constant depending solely on n. Integrating the above inequality over r from R to $2R$ and dividing by R^n we obtain

$$\left\{\frac{1}{\operatorname{meas} B_R(x)} \int_{B_R(x)} |g(y)|dy\right\}^{(n-1)/n} \leq \frac{c}{\operatorname{meas} B_{2R}(x)} \int_{R_{2R}(x)} |\operatorname{adj} Du| dy \leq cMf,$$

where Mf is the maximal function of $f = 1_\Omega|\operatorname{adj} Du|$. Thus

$$Mg(x) \leq c\{Mf(x)\}^{n/(n-1)} + C(d)\|u\|^n_{W^{1,n}}.$$

Now $f \in L^{n/(n-1)}$, and hence **[S1**, I, Theorem 1]

$$\|Mf\|_{L^{n/(n-1)}} \leq c\|f\|_{L^{n/(n-1)}} \leq c\|u\|_{W^{1,n}(\Omega)},$$

so that (4) follows.

Applications. Theorem 1 allows to sharpen previous results by Reshetnyak **[R]** and Ball **[B]** on the weak continuity of determinants.

COROLLARY 4. *Let $\Omega \subset \mathbf{R}^n$ be open and bounded, and assume that the sequence of mappings $u^{(j)}: \Omega \to \mathbf{R}^n$ satisfies* $\det Du^{(j)} \geq 0$ *and $u^{(j)} \rightharpoonup u$ (weakly) in $W^{1,n}(\Omega; \mathbf{R}^n)$. Then*

(6) $$\det Du^{(j)} \rightharpoonup \det Du \text{ (weakly) in } L^1(K),$$

for all compact sets $K \subset \Omega$.

In **[R, B]** it is shown that $\det Du^{(j)} \rightharpoonup \det Du$ weak* in the sense of measures. Since $\|u^{(j)}\|_{W^{1,n}} \leq C$, Theorem 1 in combination with the criterion on weak compactness in L^1 (see **[ET**, VIII, Theorem 1.3]) implies that the sequence $\det Du^{(j)}$ is weakly relatively compact in $L^1(K)$, and (6) follows. Corollary 4, but not Theorem 1, can also be deduced from a recent result by Zhang **[Z]**. In **[M]** Corollary 4 is used to improve a result of Ball and Murat **[BM**, Theorem 6.1] on the existence of minimizers in nonlinear elasticity. Both Theorem 1 and Corollary 4 should also have interesting applications in geometry.

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NOTE ADDED IN PROOF. Since this paper was submitted, Theorem 1 has let to several interesting developments. R. Coifman, Y. Meyer, P. L. Lions and S. Semmes found a new proof based on 'hard' harmonic analysis. Assuming only $u \in W^{1,n}$ they show first that det Du is in the Hardy space $\mathscr{H}^1$ (the predual of BMO). A standard result (similar to Lemma 3) then states that a *positive* function is in $\mathscr{H}^1$ if and only if $f \ln(2+f)$ is integrable. Their proof uses directly the divergence structure of the determinant rather than geometric estimates such as the isoperimetric inequality and thus has potential applications to more general situations.

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