I would say that this is the best book to read to get a broad feel for Loeb spaces. It has all the basic material, and a lot of examples which show just what sort of things can be done with Loeb spaces, and which cannot. There is helpful advice about which analogies between standard and nonstandard concepts are helpful and which are misleading. Those who wish to continue study of nonstandard probability theory, or who prefer less general Loeb space theory oriented specifically to continuous sample path processes, should read Keisler's monograph **[K]**, which is a development of Brownian stochastic integration and the associated differential equations in a Loeb space setting, or **[AFH-KL]**, which surveys a wide variety of applications of nonstandard analysis, with emphasis on probability theory. Of course, we warmly recommend the different departure, [N]. A good, current general introduction to nonstandard analysis is **[HL]**.

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General theory of Markov processes, by Michael Sharpe. Academic Press, San Diego, 1988, xi + 419 pp., $49.50. ISBN 0-12-639060-6

Probability theory is concerned with *random variables* and their distributions, and a family of random variables (X_t) indexed by a parameter is called a *stochastic process*. Among stochastic processes, one can roughly distinguish two main categories, whose study uses widely different methods: true stochastic processes are those for which there is indeed some process going on, i.e. for which t can be interpreted as *time*, the order structure of the index set (possibly the integers, or some interval of the line, or more recently a quadrant of $\mathbf{R}^n$) playing a fundamental role. For the remaining ones "t" has no temporal meaning at all (it may denote an element of a group, of a topological space...). Many processes of the latter category are now called *random fields* rather than stochastic processes. I don't mean the distinction is always clear, but let us accept it for the sake of simplicity.

The basic ideas in "true" stochastic processes theory can already be found in Doob's book *Stochastic processes* (1953). One associates with every "time" t a σ-field $\mathcal{F}_t$ which represents our knowledge at time t (this is nowadays called a *filtration*). One then defines *stopping times*, which are random times T such that for all t $(T \leq t) \in \mathcal{F}_t$, meaning that one doesn't need to "look into the future" to know the value of T (there is a deep difference between the apparently similar statements "when did you start smoking" and "when did you stop smoking": at the time you smoked your last cigarette, you couldn't be sure it was really the last). One of the basic principles of the theory is "to do for stopping times everything that can be done at constant times." A third feature is the constant use of *martingale theory*, which appears naturally when studying the best prediction at time t of some fixed random variable Z, which is generally taken to be the conditional expectation $\mathbf{E}[Z|\mathcal{F}_t]$. Doob's results on the regularity properties of martingales, mixed with Ray-Knight compactification methods, are one of the basic techniques of proof in this book.

The simplest situation from the point of view of prediction is that where observing the value at time t of some stochastic process (X_t) (taking values in some state space E) allows to do the best possible prediction of all future events, without keeping any files to record the past. Otherwise stated, $\mathbf{E}[Z|\mathcal{F}_t] = \mathbf{E}[Z|X_t]$ for all future random variables Z at all times t. This is the idea of a *Markov process*. Then all predictions can be deduced from one single analytical object, the *transition function* $P_{s,t}(x, A)$ of the process, which tells you the probability, knowing that $X_s = x$, to find X_t in a given set A at some future time t. In the cases of interest in this book, the process is *time homogeneous*, the transition function depending only on the difference $t - s$ and the basic object being a *transition semigroup* (P_t).

Markov processes are a simplification of general stochastic processes, but the interplay between Markov processes and general processes is very subtle, and has been historically fruitful. For Markov processes explicit computations can be done using the semigroup, and provide many examples and counterexamples. The usual Markov processes are simple objects, but "general" Markov processes can be as complicated and subtle as anything in the world—and in fact, even the simplest looking Markov processes, i.e. Markov chains on a countable state space, can be devilishly complicated as soon as hypotheses are relaxed a little. The "general" theory of stochastic processes has greatly benefited from Markov processes, since several of its main ideas, like the so called predictable processes, the decomposition of supermartingales, and local times, have been first tried on Markov processes (and the first one, which now looks very basic, is a by-product of results of K. L. Chung [**1**] on Markov chains). On the other hand, the theory of Markov processes itself uses a lot of "general" theory (in this book, stochastic integration), and also an unusual amount of measure theory, certainly a serious nuisance to its popularity.

The general theory of Markov processes has been developed in parallel by the Russian school (Dynkin's famous seminar in Moscow: see [**1**] where much of Dynkin's work has been collected) and by the Western school, whose classics are Doob's papers on the relations between Markov processes and potential theory [**1, 2**], Blumenthal's paper [**1**], and Hunt's celebrated paper in three parts [**1**]. The two schools differed vastly in spirit. For instance, the Russians often insist on general nonhomogeneous transition functions instead of semigroups, and consider the transition function as something which has to be defined *from the process*, while in the West the central object seems to be the *semigroup* itself, which on the one hand is "realized" probabilistically as a Markov process, and with which, on the other hand, are associated several analytical objects, like the resolvent (i.e. the Laplace transform $U_\lambda = \int_0^\infty e^{-\lambda t} P_t\, dt$ of the semigroup which is a smoother object than (P_t)), the infinitesimal generator, etc. The work by Hunt and its followers has been concentrated twenty years ago in the well-known book by Blumenthal-Getoor [**1**].

At that time, the theory of Markov processes was very popular among probabilists: the relations between it and potential theory had been very exciting, and had made probability theory respectable among mathematicians. However, there was a general feeling that prosperity couldn't be eternal, and that the general theory of Markov processes would slowly cool down and become a dead star. This prophecy turned out to be wrong. It is true that most of the activity about stochastic processes (in the sense defined at the beginning) has moved to neighbouring fields: for instance, diffusion theory and its relations with stochastic differential equations and geometry, large deviations, and Malliavin's striking use of infinite dimensional analysis. On the other hand, a relatively small group of mathematicians (among which the author himself) has kept the theory alive and blooming over these twenty years, and the recent developement doesn't give signs of a failing health.

Most of this book describes fairly recent progress. The central one is the understanding of *what a "good" Markov process really is.* It all started with a short paper of C. T. Shih [**1**], showing that Hunt's balayage theorem (whose content is roughly that the infimum of all excessive (=superharmonic) functions dominating a given excessive function f on a set A can be interpreted as the expectation of $f(X_T)$ at the time T the process first reaches A) has a much wider scope than it was believed, and can be extended essentially to all Markov processes by a method of compactification. This led to the definition of the so called *right processes* (Meyer-Walsh [**1**], Getoor [**1**]) which made obsolete the old categories of *Hunt processes*, *standard processes*, etc. In another way, this can be considered a triumph of the compactification methods due to Ray [**1**], Knight [**1**], and (a long way back in time) first introduced in Markov chain theory as "fictitious states."

Another central idea which made this way during the twenty years 1968–1988 is the role of *homogeneity* in the theory, thanks to an important paper of Azéma **[1]**. In the theory of stochastic processes we have a filtration $(\mathcal{F}_t)$ representing the past, but there is nothing to represent the future. It turns out that the best way to represent the future is a family of *shift operators* (B_t) (an idea which can be traced back to Dynkin), events which lie in the future w.r.t. time t being defined as arbitrary events shifted by t. Once shift operators B_t have been introduced (with the property that $B_{s+t} = B_s B_t$) one can define *homogeneous processes*, and *homogeneous random measures*. A homogeneous random measure which is finite on finite intervals can be (like an ordinary measure on the line) characterized by its (random) distribution function, which is an *additive functional*: calling $A_t(\omega)$ the mass that the measure ascribes to the interval $]0, t]$, the homogeneity of the measure is translated into the identity $A_{t+s}(\omega) - A_s(\omega) = A_t(B_s\omega)$. Such additive functionals have been known for a long time, and given a measure μ on the state space, one can associate to it an additive functional which roughly tells "how much" of the measure μ is seen along each sample path of the process. This has become the modern theory of *Revuz measures* **[1]**, well represented in this book.

But a feature of recent times is the importance given to homogeneous random measures which do not give finite mass to intervals $]0, t]$, for instance the measure which counts the jumps of a Lévy process, or which counts end points (left or right) of zero free intervals of brownian motion. The study of such measures is one of the keys to the local study of sample paths, and much space is devoted to it in this book. Applications are given to a general version of excursion theory (developed by Maisonneuve **[1]**, but also in a remarkable paper by Dynkin **[1]** which had too little influence in the West) and to the theory of Lévy systems.

This book is written with extreme care, and will remain one of the basic references on a number of subjects. On the other hand, it doesn't cover all of the "general" theory, even in its classical forms. The most notable exception is *duality* between Markov processes, in all its aspects, probabilistic (time reversal of a Markov process) or analytic (the theory of Martin compactifications and Martin boundaries, which is the Ray-Knight compactification theory, well developed in this book, but applied to the dual semigroup). Another subject which hasn't found room into the book is the very fashionable one of *Kuznetsov measures* (of which a very useful particular case was independently discovered by J. Mitro **[1]**). Since much of the author's own work concerns these subjects, we may regret these omissions, but then the book would have taken ten years more to be written. So let us fix an appointment twenty more years from now, to the year 2008.

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Approximation of continuously differentiable functions, by J. G. Llavona. North-Holland Mathematics Studies, vol. 130, North-Holland, Amsterdam, 1986, xiv + 241 pp., $48.00. ISBN 0-444-70128-1

The year 1885 was an important year for approximation theory, for in that year Weierstrass and Runge announced well-known approximation theorems bearing their names. It is the 1885 theorem of Weierstrass, asserting the density of polynomials in the real variable in the Banach space $C[a, b]$ where $[a, b]$ is a closed interval, that will concern us in this review. Since then several important extensions of the theorem have been obtained by De la Vallé Poussin **[17]**, Bernstein **[5]**, Stone **[15]**, and Whitney **[18]** and others, by stressing one aspect or another of the classical