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Toposes and local set theories: An introduction, by J. L. Bell. Oxford Logic Guides: 14, Clarendon Press, Oxford, 1988, xii + 267 pp., $75.00. ISBN 0-19-853274-1

It has recently become evident that two apparently different formulations of the foundations of mathematics are merely opposite sides of the same coin. The first of these is the theory of types, going back to Russell and Whitehead in their monumental attempt to rescue Frege from paradox, while the second is the theory of categories, invented by Eilenberg and Mac Lane and conceived as the appropriate language for the foundations of mathematics by Lawvere.

The theory of types, or higher order logic, is called *local set theory* by Bell. As he puts it "types may be thought of as *natural kinds* or *species* from which sets are extracted as subspecies. The resulting theory of sets is *local* in the sense that, for example, the inclusion relation will only obtain among sets which have the same type...."

Unfortunately, the original type theory in Principia Mathematica had proved too cumbersome for most people and, in spite of more elegant formulations by Church and Henkin, was replaced by the set theories of Gödel-Bernays, favoured by mathematicians, and Zermelo-Fraenkel, favoured by logicians. However, in these languages one can ask such meaningless questions as whether the Klein four-group is included in π.

The following simple presentation of type theory had been proposed by Phil Scott and the reviewer **[LS 1983]**. There are given three basic types:

$1 =$ a specified one-entity type introduced for convenience,

$\Omega =$ the type of truth-values or propositions,

$N =$ the type of natural numbers.

From these other types are built up by two operations:

$A \times B =$ the type of pairs of entities of types A and B,

$PA =$ the type of all sets of entities of type A.

In *pure* type theory there will be no other types than those in the hierarchy constructed from the three basic types by the two operations; but in *applied* type theories there may very well be other types, as we shall see later.

A type theory, pure or applied, is a formal language consisting of terms of different types. Among the terms there are countably many variables of each type; we write $x \in A$ to say that x is a variable of type A. From the variables other terms are defined inductively as follows:

$$\begin{array}{ccccc} 1 & \Omega & N & A \times B & PA \\ * & a = a' & 0 & \langle a, b \rangle & \{x \in A \mid \varphi(x)\} \\ & a \in \alpha & Sn & & \end{array}$$

where it is assumed that a and a' are terms of type A already constructed, α of type PA, n of type N, b of type B and $\varphi(x)$ of type Ω. The usual logical connectives and quantifiers may now be defined, for instance:

$$T \equiv * = *,$$
$$p \wedge q \equiv \langle p, q\rangle = \langle T, T\rangle,$$
$$p \Rightarrow q \equiv p \wedge q = p,$$
$$\forall_{x\in A}\varphi(x) \equiv \{x \in A | \varphi(x)\} = \{x \in A | T\}.$$

Bell does essentially the same thing; except that, in place of 1 and $A \times B$, he has $A_1 \times \cdots \times A_n$, with $n \geq 0$. This allows him to incorporate function symbols $f\colon A_1 \times \cdots \times A_n \to B$. To this reviewer, these function symbols appear to be analogous to Gentzen's sequents or Bourbaki's multilinear mappings; they may in fact be viewed as operations in a multisorted algebraic theory.

After the grammar of type theory has been presented, it remains to state the axioms and rules of inference. This is usually done in terms of a deduction symbol $\vdash$. Of course, one should be able to infer the usual rules of universal specification and existential generalization, in particular,

$$(1) \qquad \forall_{x\in A}\varphi(x) \vdash \varphi(x), \qquad \varphi(x) \vdash \exists_{x\in A}\varphi(x).$$

From these one would expect to infer, by transitivity of deduction, that

$$(2) \qquad \forall_{x\in A}\varphi(x) \vdash \exists_{x\in A}\varphi(x).$$

In an applied type theory, a difficulty would now arise when A is an "empty" type, for example the type of unicorns. Modern mathematicians would accept the statement

(a) all unicorns have horns,

but not

(b) some unicorns have horns.

Aristotle would have got around the difficulty by denying (a); but we cannot do so, as long as we paraphrase $\forall_{x\in A}\varphi(x)$ as $\forall_x(x \in A \Rightarrow \varphi(x))$.

In **[LS 1980]** we used the device of putting a subscript X, denoting a finite set of variables, on the deduction symbol. We wrote

$$p_1, \ldots, p_n \vdash_X p_{n+1}$$

only when all the variables occurring freely in the p_i are elements of X. The symbol $\vdash_X$ is subject to two structural rules, in addition to the usual ones of Gentzen:

$$(3) \qquad \frac{p_1, \ldots, p_n \vdash_X p_{n+1}}{p_1, \ldots, p_n \vdash_{X\cup\{x\}} p_{n+1}}, \qquad \frac{\varphi_1(x), \ldots, \varphi_n(x) \vdash_{X\cup\{x\}} \varphi_{n+1}(x)}{\varphi_1(a), \ldots, \varphi_n(a) \vdash_X \varphi_n(a)},$$

where it is assumed that a is of the same type as x and contains free occurrences of only such variables as are elements of X.

If variables are thus declared, (1) should be replaced by

$$(1') \qquad \forall_{x\in A}\varphi(x) \vdash_x \varphi(x), \qquad \varphi(x) \vdash_x \exists_{x\in A}\varphi(x).$$

From this it then follows by transitivity of $\vdash_x$, a special case of Gentzen's cut, that

$$(2') \qquad \forall_{x\in A}\varphi(x) \vdash_x \exists_{x\in A}\varphi(x).$$

From (2′) we may infer (2) according to (3), provided we can substitute a closed term a of type A for x. But when A is "empty," there are no closed terms of type A; our language does not contain the names of entities which do not exist.

I was surprised to see that Bell does not put a subscript on his deduction symbol to declare the variables which are allowed to occur freely. So how does he get around the nonexistence of horned unicorns? By denying the transitivity of deduction and Gentzen's cut, except in special circumstances! In fact, his restrictions don't allow him to infer (2) from (1) if x is free in $\varphi(x)$, since it is free in neither $\forall_{x\in A}\varphi(x)$ nor in $\exists_{x\in A}\varphi(x)$. Still, he could have inferred (2) from (1) if x is not free in $\varphi(x)$; but in that case he would have denied (1) in the first place. In other words, he does not accept $\forall_{x\in A}p \vdash p$ if p does not contain x! Strange as this interlocking set of restrictions seems at first sight, it appears to do the job and presumably occurs in the literature and has been tested by time. Personally, I believe in transitivity of deduction and, to the best of my recollection, proposed the device of declaring variables in Oberwolfach in 1974, where I was told that I had been anticipated by Mostowski.

Type theory as presented here suffices for arithmetic and analysis, although not for category theory and modern metamathematics, provided one carries out the reductionist program of Frege et al. and takes the trouble to define all the concepts needed in terms of the basic symbols $*$, $=$, $\in$, 0, S, $\{\ |\ \}$ and $\langle\ ,\ \rangle$.

However, there is another way of presenting the foundations of mathematics, essentially due to the vision of Bill Lawvere. Let me quote Bell, who puts it very elegantly: "In category theory the morphisms (arrows) between structures (objects) play an autonomous role which is in no way subordinate to that played by the structures themselves. So category theory is like a language in which the 'verbs' are on equal footing with the 'nouns'." Lawvere's earlier attempt [1964] to axiomatize the category of sets led ultimately to the introduction of "elementary" toposes by Lawvere **[1972]** and Tierney **[1972]**, generalizing the "Grothendieck" toposes **[GV 1972]** which were being used in algebraic geometry.

A *topos* is a category with finite cartesian products, a subobject classifier Ω, power objects PA and, usually also, a natural numbers object N. The subobject classifier is to allow us to mimic the usual one-to-one correspondence between subsets of A and their characteristic functions $A \to \Omega = \{T, \perp\}$, while PA is meant to be isomorphic to Ω^A, the set of all funtions $A \to \Omega$. Among the examples one has in mind are not only the usual category of sets, but also the category of sheaves on a topological space. This makes it necessary to drop the requirement that $\Omega \cong \{T, \perp\}$ in general, although this is so in what one calls a *Boolean* topos. Finally, the natural numbers object is the "least fixpoint" of the functor which sends the object A onto the coproduct of A with the terminal object 1, as computer scientists would rephrase Lawvere's original definition.

Just about everyone who thought about toposes came up with the observation that a topos has an *internal language*, a version of applied type theory, though the first to publish a description of this language for Boolean toposes was Mitchell **[1972]**. One thinks of the objects of the topos as types and of the arrows $1 \to A$ as terms of type A, hence the possibility of "empty" types when $\mathrm{Hom}(1, A) = \phi$. In particular, arrows $1 \to \Omega$ are formulas and arrows $1 \to N$ are numerals. Moreover, $p_1, \ldots, p_n \vdash q$ is taken to mean: if $p_i = T$ for $i = 1, \ldots, n$ then $q = T$. Here $T \colon 1 \to \Omega$ is a distinguished arrow, implicit in the description of Ω as subobject classifier.

In a Boolean topos one has $\vdash \forall_{x \in \Omega}(x \vee \neg x)$, but in general this is not so. Thus, the internal language of a topos turns out to be intuitionistic! This, I believe, came as a surprise to the founders of topos theory, as they had not been motivated by the philosophy of Brouwer, but rather by the philosophy of Heraclitus and his modern disciples.

On the other hand, given an intuitionistic type theory $\mathscr{L}$, one can construct the *topos generated* by $\mathscr{L}$, exactly as a nominalist would construct the category of sets from the theory of types. Thus, a *set* α of type PA is just a closed term of type PA modulo provable equality. A *function* from α to β, say of type PB, is a set ρ of type $P(A \times B)$ for which one can prove that $\rho \subseteq \alpha \times \beta$ and that

$$\forall_{x \in A}(x \in \alpha \Rightarrow \exists !_{y \in B} \langle x, y \rangle \in \rho).$$

These "sets" are the objects and these "functions" are the arrows of the topos generated by $\mathscr{L}$. The first to publish this construction was Hugo Volger **[1975]**, although it probably occurred to many people independently. It was also noticed that the topos generated by the internal language of a topos $\mathscr{T}$ is equivalent to $\mathscr{T}$. Thus every topos is equivalent to a *linguistic* topos, justifying a nominalistic view of mathematics.

More can be said: if $L(\mathscr{T})$ is the internal language of the topos $\mathscr{T}$ and $T(\mathscr{L})$ is the topos generated by the type theory $\mathscr{L}$, we may extend L and T to functors between two categories, the category of (small) toposes and the category of (small) type theories. The morphisms of the former category are well known, they are the *logical functors* which preserve everything on the nose. Morphisms of the latter category were called *translations* in **[LS 1986]**, they are meant to send types to types and function symbols to function symbols. As we pointed out in a series of exercises, a different choice of morphisms would have yielded an equivalence of categories; but this was not needed, as all our applications could be obtained from the fact that T is left adjoint to L.

What this means is that there is a natural one-to-one correspondence between the translations $\mathscr{L} \to L(\mathscr{T})$ and the logical functors $T(\mathscr{L}) \to \mathscr{T}$. Unfortunately, this statement requires some handwaving, unless we tighten the definition of "topos" somewhat: we insist that toposes possess "canonical" subobjects and that logical functors preserve them. If this seems unnatural at first sight, it should be pointed out that all toposes occurring in nature do have canonical subobjects, as do all linguistic toposes, and we recall that every topos is equivalent to a linguistic one. Bell wisely prefers not to bother with tightening the definition of topos and proving adjunction, although perhaps he could have done this cheaply by confining himself to linguistic toposes.

Either of the arrows $\mathscr{L} \to L(\mathscr{T})$ or $T(\mathscr{L}) \to \mathscr{T}$ may be taken to be an *interpretation* of $\mathscr{L}$ in $\mathscr{T}$. Bell wishes to define this term at an earlier stage, before having discussed the morphisms of the two categories involved. This obliges him to give a rather tedious inductive definition. He establishes soundness and completeness: a statement in a type theory $\mathscr{L}$ is provable if and only if it holds under every interpretation in a topos. In particular, a statement in pure intuitionistic type theory, ostensibly about sets, also holds for sheaves in place of sets. This added generality is the reward one reaps for doing mathematics constructively.

This reviewer would prefer to attach the word "completeness" to a more difficult theorem, an intuitionistic generalization of the Gödel-Henkin completeness theorem **[H 1950]** of higher order logic. An interpretation of a type theory in a topos is to be regarded as a *model* only if the topos shares the following three properties with the category of sets:

(1) not every proposition holds;

(2) if $p \vee q$ holds then p holds or q holds;

(3) if $\exists_{x\in A}\varphi(x)$ holds then $\varphi(a)$ holds for some arrow $a\colon 1 \to A$ in the topos.

As was first observed by Peter Freyd **[1978]**, these three properties of a "model" topos have algebraic translations concerning the terminal object 1:

(1) 1 is not initial;

(2) 1 is indecomposable;

(3) 1 is projective.

A fairly deep result, mentioned but not proved in Bell's book, asserts that the so-called *free topos*, the topos generated by pure type theory, an initial object in the category of all small toposes, is a model topos. This implies for pure type theory:

(1) $\perp$ is not provable;

(2) if $\vdash p \vee q$ then $\vdash p$ or $\vdash q$;

(3) if $\vdash \exists_{x\in A}\varphi(x)$ then $\vdash \varphi(a)$ for some term a of type A.

While every mathematician believes in (1), (2) and (3) fail in classical type theory, as follows from Gödel's famous incompleteness theorem **[1931]**. Yet intuitionists have always believed in (2) and (3), even before these assertions had been proved as metatheorems.

The free topos, like every model topos, may be viewed as a "possible world," acceptable to moderate intuitionists. Being an initial object, it may justly be called "the best of all possible worlds," which Platonists might view as the "real world."

The internal language of a topos may be exploited to prove categorical properties of the topos linguistically. Bell does this even for properties which other people have included in the definition of a topos, such as being cartesian closed, finitely complete and cocomplete.

To a type theory $\mathscr{L}$ one easily adjoins a "parameter" x of type A to obtain the type theory $\mathscr{L}(x)$ whose closed terms are terms of $\mathscr{L}$ with no free occurrences of variables other than x. The corresponding construction for toposes is equally simple: the *slice* category $\mathscr{T}/A$ has as objects arrows $C \to A$, where C ranges over objects of $\mathscr{T}$, and as arrows the obvious commutative triangles. The connection between these two constructions is that $\mathscr{T}/A$ is equivalent to $T(L(\mathscr{T})(x))$. Thus $\mathscr{T}/A$ may be thought of as the result of adjoining an *indeterminate arrow* $x\colon 1 \to A$ to $\mathscr{T}$, much as one adjoins an indeterminate element to a commutative ring, as was first noticed by Joyal in this generality.

With any arrow $f\colon A \to B$ there is associated a translation $\tau_f\colon \mathscr{L}(y) \to \mathscr{L}(x)$, where y is a variable of type B, which sends any term $\varphi(y)$ of $\mathscr{L}(y)$ onto the term $\varphi(fx)$ of $\mathscr{L}(x)$. The corresponding logical functor $f^*\colon \mathscr{T}/B \to \mathscr{T}/A$ happens to have both a right and a left adjoint. When $\varphi(y)$ is of type Ω, one says that B *forces* $\varphi(f)$, or that f *satisfies* φ at stage B, to mean that $\forall_{x \in A}\varphi(fx)$ holds in $\mathscr{T}$. This is the categorical version of Paul Cohen's forcing relation. When defined inductively on the complexity of φ, it gives rise to Joyal's version of Beth-Kripke **[1965]** semantics for intuitionistic logic.

As we have seen, topos theory was born from a union between logic and geometry. Just as logical functors were inherited from one parent, so geometric morphisms were inherited from the other. The latter were completely ignored in **[LS 1986]**, but are covered extensively by Bell. The motivation is this: given a continuous function $f\colon X \to Y$ between topological spaces, one constructs a functor $f_*\colon \mathrm{Sets}^{X^{\mathrm{op}}} \to \mathrm{Sets}^{Y^{\mathrm{op}}}$ as follows: for any presheaf $P\colon X^{\mathrm{op}} \to \mathrm{Sets}$ define the presheaf $f_*(P)\colon Y^{\mathrm{op}} \to \mathrm{Sets}$ on any open subset V of Y by $f_*(P)(V) = P(f^{-1}(Y))$. It turns out that f_* has a left adjoint f^*, which moreover is left exact, that is, which preserves finite products and equalizers.

Quite generally, if $f_*\colon \mathscr{T} \to \mathscr{T}'$ is any functor between toposes which possesses a left exact left adjoint, one calls f_* a *geometric morphism*. In particular, the inclusion functor from $\mathrm{Sh}(X)$, the category of sheaves on X, to $\mathrm{Sets}^{X^{\mathrm{op}}}$, the category of presheaves on X, is a geometric morphism, its left adjoint being called *sheafification*. This process was generalized by Grothendieck, who introduced something called a "topology" on any small category $\mathscr{C}$, allowing him to obtain in the same way a full reflective subcategory of $\mathrm{Sets}^{\mathscr{C}^{\mathrm{op}}}$, called a *Grothendieck topos*, such that the reflector is left exact. In fact, every such subcategory of $\mathrm{Sets}^{\mathscr{C}^{\mathrm{op}}}$ can be obtained in this way.

There is an interesting analogy with module categories, where the corresponding reflectors are called "localizations" and the corresponding "topologies" were introduced by Gabriel. The latter are certain filters of right ideals of the ring. They are in one-to-one correspondence with (hereditary) torsion theories. Moreover, "torsionfree modules" correspond to "separated presheaves" and "torsionfree divisible modules" correspond to sheaves.

One can play a similar game in elementary toposes in place of functor categories. There the topologies are determined by arrows $j\colon \Omega \to \Omega$, called *modalities* by Bell, which satisfy the following conditions:

$$x \vdash_x jx,\; jjx \vdash_x jx,\; \frac{x \vdash_{\{x,y\}} y}{jx \vdash_{\{x,y\}} jy},$$

where x and y are variables of type Ω. He thinks of j as a kind of "possibility" operator.

Bell proposes an interesting and original analogy with the theory of relativity. He views a geometric morphism between two toposes as something akin to a coordinate transformation, which may serve to simplify the description of some phenomena. For example, consider a continuous real-valued function f on a topological space X. In the category of sets, $f(x)$ may be viewed as a real number varying continuously over X. However, in the topos $\mathrm{Sh}(X)$, everything varies over X, so the variation of $f(x)$ is not noticed and f becomes a constant real number. "The concept of 'real number,' interpreted in $\mathrm{Sh}(X)$ corresponds to the concept of 'real-valued continuous function on X' interpreted in Set."

To pursue a related analogy discussed by Bell, recall that Maxwell's equations have been formulated so as to be invariant under change of coordinate system, while Ohm's law has not. In the same way, constructively provable statements are valid in any topos, but the law of excluded middle is not. Although this law is preserved by logical functors, it is not preserved by geometric morphisms.

The first order formulas which are preserved by geometric morphisms are called *geometric implications*. They have the form

$$\forall_{x_1} \cdots \forall_{x_n} (\varphi(x_1, \ldots, x_n) \Rightarrow \psi(x_1, \ldots, x_n)),$$

where φ and ψ may contain $\wedge$, $\vee$ and $\exists$ but not $\Rightarrow$ and $\forall$. It has been known **[MR 1976]** that any geometric implication which has a classical proof also has an intuitionistic one. Bell gives an interesting, but quite simple, proof of this, using the following theorem by Barr **[1974]**, for which the reader is referred to the book by Johnstone **[1977]**: for every Grothendieck topos $\mathscr{G}$ there is a Boolean topos $\mathscr{B}$ and a geometric morphism $f_*: \mathscr{B} \to \mathscr{G}$ such that f^* is faithful. I am told that this is the application Lawvere had in mind when he suggested the theorem to Barr.

Having co-authored a monograph on a very similar topic, I approached this review with some apprehension. I need not have worried; Bell confirms most of our views and adds a number of new insights. He covers a lot of material that we did not, not all of which has been touched in this review. His book is a delight to read and I would recommend it to my students as well as to mathematicians at large.

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J. LAMBEK
MCGILL UNIVERSITY

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Information-based complexity, by J. F. Traub, G. W. Wasilkowski and H. Woźniakowski. Academic Press, Boston, San Diego, and New York, 1988, xiii + 523 pp., $64.50. ISBN 0-12-697545-0

In the past ten years, there have been a good number of developments in information-based complexity theory; in addition, the field and related issues have gained more attention in the mathematical community. This book fills a need for information on recent developments, and it comprehensively describes older and better-known results.