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Rational approximation of real functions, by P. P. Petrushev and V. A. Popov. Encyclopedia of Mathematics and its Applications, vol. 28, Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne and Sydney, x + 371 pp., $69.50. ISBN 0-521-33107-2

A book on "approximation theory" can deal with almost any topic. Even "rational approximation" leaves a vast field of mathematics including famous theories like the characterization of those compacts in the complex plane on which good functions can be well approximated by rationals (the Vitushkin theorem, etc.). But *Rational approximation of real functions* makes it quite clear that we are concerned with an important part of the constructive theory of functions in the sense it has been developed mainly by Soviet mathematicians in the tradition of S. N. Bernstein. The book of almost 400 pages by Petrushev and Popov under review gives a stimulating account of the development up to the most recent time of that field, to which both authors have made significant contributions.

The word real in "real functions" is not taken too seriously. It is certainly not easy to completely avoid the complex if you wish to include a chapter on Padé approximation. There is however another reason to involve complex thinking in every discussion of rational approximation (and a very important one according to the reviewer's opinion). Sooner or later you have to comment upon the question if and why rational approximation is "better" than polynomial. The answer is that the rationals are better if the information about the function that shall be approximated is such that you can benefit from your freedom to choose the poles (the pole of a polynomial is quite fixed!). That is true (cf. [**Ga**, Chapter 3]) in the natural generalizations of Zolotarjov's problems from around 1870. (These problems are discussed in the fourth chapter of the book under review.) That is also "the explanation" of the interesting difference between polynomial and rational approximation: How come that in rational approximation the best estimate in the case of a fixed function in some natural class is better than the estimate for the class? The typical example is the following, conjectured by Donald Newman and proved by V. A. Popov.

Let $R_n(f)$ denote the best approximation in the uniform norm by rationals of order n of the function f and let Lip 1 denote the Lipschitz class. Then, $f_1 \in \mathrm{Lip}\,1$ implies that $R_n(f_1) = o(n^{-1})$ but $\sup_{f\in \mathrm{Lip}\,1} R_n(f) \neq o(n^{-1})$.

This is one example of Newman's contributions that revived rational approximation in the sixties. His most famous result is the discovery [**N**] that $|x|$ on $[-1, 1]$ can be uniformly approximated by rationals within $\exp(-c\sqrt{n})$, while the best polynomial approximation is $O(n^{-1})$, not even $o(n^{-1})$.

When G. G. Lorentz in 1966 gathered topics for a neat little treatise on approximation theory, he devoted 5 pages out of 190 to rational approximation, the only direct theorem being Newman's just mentioned. The activity in the field since then is well documented by Petrushev and Popov in the book under review with 120 theorems and more than 300 references. But there is a long history of the subject as is clear from the name of Zolotarjov mentioned above.

The true joy for a rational approximator is when the crucial step in the proof can be a reference to some classical, less computational, theorem like Chebyshev's or de La Vallée-Poussin's. Alas that is not the case every time, and that makes the writing of a book like *Rational approximation of real functions* a difficult task. Books should certainly be written about general methods and principles and not be filled with clever tricks, but much of the progress in approximation theory has been by such tricks. Many of us have looked forward to study a long-awaited proof of a theorem that has intrigued us, in the hope of learning something useful to apply in adjacent parts of the field. That that often turns out to be a deception is probably true also in other parts of mathematics.

I admire the authors for their skill in creating a balance between an emphasis on fundamental principles and the tedious computations which are necessary in the proofs of many theorems. The treatment of measures of continuity and smoothness is an example: the approach to direct and converse theorems by application of Peetre's K-functional after having established the appropriate Jackson and Bernstein inequalities gives a unified treatment of approximation in several important spaces.

Besides the parts devoted to general theory and approximation by polynomials and rationals, there are chapters on splines (in fact about one fourth of the book), Padé approximation and approximation with respect to Hausdorff distance. The main part of the chapter on Hausdorff approximation deals with rational approximations of sign x and supports the authors' claim that this is the way to understand the bound $\exp(-c\sqrt{n})$. The theorems on Padé approximation are almost all due to or connected with A. A. Gonchar.

In a chapter on the approximation of some important functions we are informed about the solution of three problems, which stayed open for some time. First we get the proof of N. S. Vjacheslavov's sharp estimate for Newman's $|x|$-problem: On $[-1, 1]$ the best rational approximation $R_n(|x|)$ has the property that $R_n(|x|)\exp(\pi\sqrt{n})$ is bounded from above and below by positive constants (proof published in 1975). The second problem is Meinardus' conjecture on the best uniform approximation by rationals to $\exp x$ on $[-1, 1]$. The conjecture was proved by D. Braess in 1984 by an elegant method applying, among other things, tools from Padé approximation.

The last problem, and probably the most important, is the limit of $R_n^{1/n}$ where R_n is the best uniform rational approximation of $\exp(-x)$ on the half-line $[0, \infty)$. Gonchar and Rahmanov **[GoR]** proved in 1986 that the limit exists and can be expressed by elliptic integrals, but the proof is not given in the book.

It might be allowed to mention that there is quite a number of misprints, but most of them are quite harmless like misspelling of names. It is more regrettable that the first definition of Hausdorff distance is blurred by a missing comma. That observation does not influence my opinion that this is a very good book for anyone interested in constructive function theory and that it certainly can be used as an educative graduate text-book.

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Introduction to the spectral theory of polynomial operator pencils, by A. S. Markus. Translated by H. H. McFaden, Translations of Mathematical Monographs, vol. 71, American Mathematical Society, Providence, R.I., 1988, iv + 250 pp., $95.00. ISBN 0-8218-4523-3

This book concerns the spectral theory of operator polynomials, i.e., of expressions of the form

$$A(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^n A_n, \tag{1}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and $A_0, \ldots, A_n$ are operators acting in a Hilbert space H. This subject also includes the spectral theory of a single operator, which appears if one takes $n = 1$. The motivation for a polynomial spectral theory arises in the study of differential equations

$$\begin{cases} A_n\varphi^{(n)}(t) + \cdots + A_1\varphi^{(1)}(t) + A_0\varphi(t) = 0, \\ \varphi^{(j)}(0) = x_j, \qquad j = 0, \ldots, n-1. \end{cases} \tag{2}$$

Here the unknown function φ is a H-valued function on $0 \le t < \infty$ and the initial data $x_0, \ldots, x_{n-1}$ are vectors in H. A solution φ of (2) is called *elementary* whenever φ is of the form

$$\varphi(t) = e^{\lambda_0 t}\left(\sum_{\nu=0}^{k-1} \frac{1}{\nu!} t^\nu x_{k-1-\nu}\right). \tag{3}$$

If $H = \mathbb{C}^m$ and the leading coefficient A_n is invertible, then any solution of (2) is a linear combination of elementary solutions. As is well known, the latter statement follows from the general fact that for $A_n = I$ (the identity operator on H) the function (3) is a solution of (2) if and only if

$$(A - \lambda_0)\hat{x}_0 = 0, \qquad (A - \lambda_0)\hat{x}_j = \hat{x}_{j-1}, \qquad j = 1, \ldots, k-1,$$