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Lagrange and Legendre characteristic classes, by V. A. Vassiliev. Advanced Studies in Contemporary Mathematics, vol. 3, Gordon and Breach, New York, 1988, x + 268 pp., $98.00. ISBN 2-88124-661-3

1. Caustics and lagrangians. Originally, caustics were envelopes of families of light rays. If the light propagates in $\mathbf{R}^n$ from a hypersurface (the source) S_0 along the half lines γ_q normal to S_0, we can consider the set of these light rays as a submanifold L of the tangent bundle $T\mathbf{R}^n$,

$$L = \{(\gamma_q(t), p) | t \geq 0, q \in S_0, \|p\| = 1, p \in (T_q S_0)^{\perp}\}.$$

The envelope of the rays is obtained very simply from L. It is just the image of the singular points of the projection from L onto $\mathbf{R}^n$

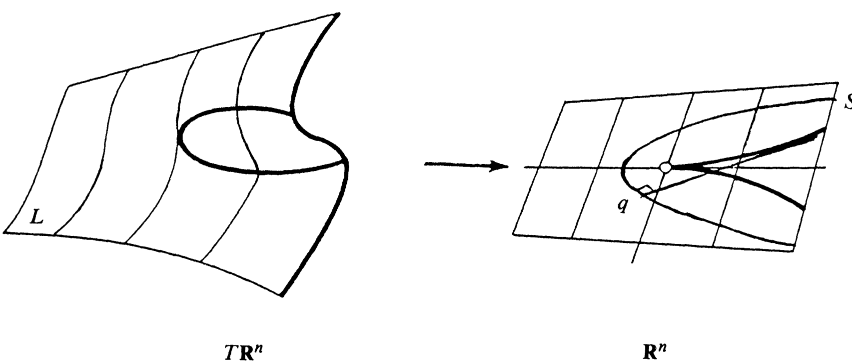

FIGURE 1.

In addition to having the same dimension as $\mathbf{R}^n$, the submanifold L has, by definition, another property: The 1-form $\lambda = p \cdot dq$ is exact when restricted to L. (The *optic length* t is a primitive of it.) It is one of the principles of geometric optics that traversing an optical system (lenses, clouds, mirrors, cups of coffee,...) induces a *canonical transformation*: The new manifold of light rays, if it is no longer the manifold of rays orthogonal to a hypersurface, retains these two properties. (See, for example, **[8]**.)

In order to generalize these two properties, we replace $\mathbf{R}^n$ by a manifold X and we dispense with the metric, replacing the tangent bundle by the cotangent bundle $T^*X \xrightarrow{\pi} X$, the form λ becoming the celebrated *Liouville form* $\lambda = pdq$: If $(q_1, \ldots, q_n)$ are local coordinates on X and $(p_1, \ldots, p_n)$ are the dual coordinates, then $\lambda = \sum_{i=1}^n p_i \, dq_i$.

We consider immersions $f : L \to T^*X$ of manifolds of the same dimension as X such that $f^*\lambda$ is a closed 1-form. We say then that f is *lagrangian*: at each point, the tangent space to L injects as a subspace of the tangent space to T^*X that is maximally totally isotropic for the nondegenerate (symplectic) 2-form $\omega = d\lambda$,

$$d(f^*\lambda) = 0 \iff f^*\omega = 0.$$

By virtue of π, we also have a definition of caustics: the caustic $C(f)$ is the set of singular points of the projection $\pi \circ f$.

2. Wave fronts and legendre manifolds. A related notion is that of *wave front*: In the example of the luninous source S_0, each of the hypersurfaces $S_t = \{\gamma_q(t) | q \in S_0\}$ is a wave front (the surface to which the light wave has arrived at time t).

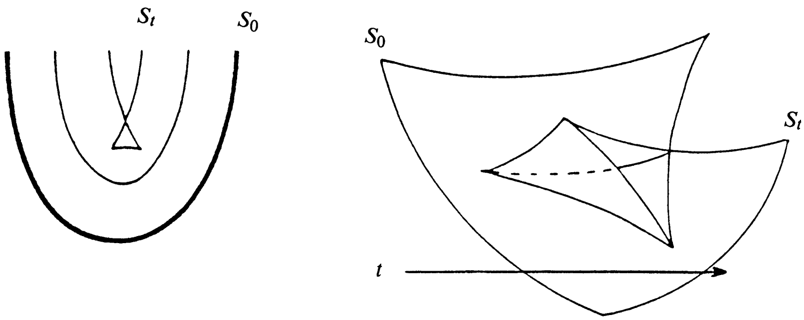

FIGURE 2.

They can have singularities, but they are the projections of smooth submanifolds of the sphere bundle $S(T\mathbf{R}^n)$,

$$L_t = \{(\gamma_q(t), p) | p \perp T_q S_0\}.$$

The L_t are smooth, and one sees clearly that the singularities that can appear on the S_t are not too complicated: At each point of S_t, there is a tangent hyperplane, the orthogonal complement of the vector p.

More generally, on the total spaces of fiber bundles such that $P(T^*X) \to X$ or $S(T^*X) \to X$ (bundles of tangent hyperplanes or of oriented tangent hyperplanes), there is a *contact structure* and the manifolds under consideration are *legendre* manifolds, maximal integrals of the contact structure. A wave front is a hypersurface of X that is the projection of an (immersed) legendre manifold.

3. Cobordisms. Rather than regard each of the wave fronts S_t separately, we can consider their union as a big front in $X \times R$ where the second factor is that of the variable t (Figure 2). Arnold **[2]** had the idea of considering this big front as a cobordism between the fronts S_0 and S_t. To do this, he defined cobordism groups for legendre immersions in certain contact manifolds. For lagrangian immersions in a cotangent bundle, he has also defined a relation of *cobordism*: Two lagrangian immersions in T^*X are cobordant if they constitute the "boundary" of a lagrangian immersion in $T^*(X \times R)$. (There is a small technical problem to define this notion because of the jump of two dimensions, but this is not very serious.) Knowing enough pieces of lagrangian surfaces, Arnold was able to "calculate" the groups thus defined in the case of curves.

In the case of oriented (lagrangian) curves in $\mathbf{R}^2 = T^*\mathbf{R}$, he showed that the group is isomorphic to $\mathbf{Z} \oplus \mathbf{R}$. One of the steps in the proof is to verify that the curve shown in Figure 3 is not a lagrangian boundary.

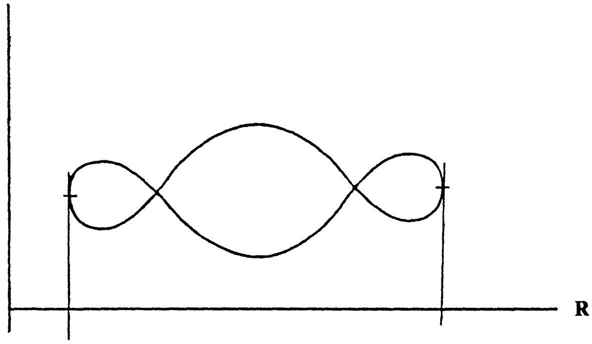

FIGURE 3.

4. The Maslov class; return to caustics. There is an invariant of lagrangian cobordism for curves: their Maslov index. Arnold had known for a long time **[1]** how to describe this in terms of the singularities of the projection onto X. The Maslov class is a cohomology class of degree 1. In the case of curves it defines an integer (the index), which is the number of singular points of the projection $\pi \circ f$ computed with appropriate signs, when this projection is sufficiently general. For example, the index of the curve in Figure 3 is ± 2, and so this curve cannot be cobordant to zero.

The work of Vassilyev which is the object of the book under review finds its source there: In all dimensions, the Maslov class is related to *folds* of the projection. It is a question of defining invariants (numbers) of lagrangian cobordism in higher dimensions using certain types of more complicated singularities of the projection, which therefore correspond to singularities of the caustic itself.

5. The generating functions of Hörmander [9][1]. It happens that the study of lagrangian singularities comes down to that of ... singularities of *functions.*

The simplest example of a generating function is the following. We consider a 1-form α on X as a section $\alpha : X \to T^*X$; its image is a copy of X embedded in T^*X. As the Liouville form has the magic property that $\alpha^*\lambda = \alpha$, the image of α is lagrangian if and only if the 1-form α is closed, that is to say, locally exact. Thus, the lagrangian that we are considering will be described locally by a function $S(q_1, \dots, q_n)$ and the equations

$$p_1 = \frac{\partial S}{\partial q_1}, \dots, p_n = \frac{\partial S}{\partial q_n}.$$

Of course, there is no question of describing, even locally, every lagrangian by such a function: The projection on X would never have singularities, but it can almost be done. For that, it suffices to replace enough of the variables q by the variables p. For example, we can describe a *fold* of the

[1]It is unfortunate that this paper of Hörmander and the paper **[5]** of Duistermaat do not figure in the bibliography of Vassilyev's book.

projection by the function

$$S(p_1, q_2, \dots, q_n) = p_1^3$$

and the equations

$$q_1 = -\frac{\partial S}{\partial p_1}, \qquad p_i = \frac{\partial S}{\partial q_i} \qquad (i \geq 2)$$

and a gather (for $n \geq 2$) by

$$S(p_1, q_2, \dots, q_n) = \pm p_1^4 + q_2 p_1^2$$

and the same equations.

These two functions appear to be essentially the universal unfoldings of singularities of types $A_2(p_1^3)$ and $A_3(\pm p_1^4)$. In a more general way, the classification of functions $(\mathbf{R}^N, 0) \to (\mathbf{R}, 0)$ with isolated singularity furnishes a classification of the singularities of lagrangian projections. Of course, all this can work only in small codimension: Here it is all the strata of codimension ≤ 7 of the space of jets that will be useful.

Thus, Vassilyev finds himself with a list of generic singularities of caustics, which are the only ones to appear in low dimensions (≤ 7). With the help of the incidence relations of the strata that contain these singularities in the space of jets of functions, he defines an[2] abstract cochain complex (C^*, δ). For every lagrangian immersion $f : L \to T^*X$, the consideration of the germ of the singularities yields a map $H(C^*, \delta) \to H^*(L)$ (for $* \leq 6$) and therefore yields certain characteristic classes and characteristic numbers for lagrange manifolds of small dimension.

By calculating the coboundary of the complex (these are calculations of the type "adjancey of singularities"), he obtains some results on the "enumerative theory" of lagrangian singularites: For example, there are "as many" points of type A_4 as of type D_4 on a closed lagrange manifold of dimension three. The calculation of the groups of lagrangian or legendrian cobordisms [3] also furnishs (indirectly) rather precise results of this kind.

6. Lagrangian characteristic classes. Every lagrangian immersion $f : L \to T^*X$ has characteristic classes in a classical sense: One verifies rather easily that f trivializes the complexification of the (stable) normal bundle $N_{\pi \circ f}$ of the projection. There exists an almost complex structure J on T^*X that pemits the identification of $T(T^*X)$ and $\pi^*TX \otimes C$. When f is lagrangian, the subspaces $T_x f(T_x L)$ and $J T_x f(T_x L)$ are transverse for every x, and we have, therefore, an isomorphism of complex vector bundles

$$(\pi \circ f)^* TX \otimes C \simeq TL \otimes C.$$

In particular, f defines a map

$$\gamma_f : L \to U/O.$$

(U/O is the classifying space for vector bundles with trivial complexifications.) The cohomology of U/O provides in return, classes in L which are called the *characteristic classes* of the lagrangian immersion f.

[2]In fact, several: There is one for each kind of contact manifold considered on X, without considering the distinction oriented/nonoriented.

The cohomology of U/O is well known: It is one of the explicit examples in Borel's thesis **[4]**, and a nice geometric description of the generators has been given by Fuks **[7]**. For example, the Maslov class is the generator of $H^1(U/O) \simeq Z$. Among these classes, those of degree equal to the dimension allow us to construct *characteristic numbers*, which are invariants of lagrange cobordism. By using the h-principle of Gromov, Eliashberg **[6]** has remarked that modulo torsion (and some nondiscrete invariants of which, for lack of space, we cannot consider here), all the cobordism invariants are of this form.

The arrow $H(C^*\delta) \to H^*(L)$ that defines the characteristic classes of Vassilyev factorizes by way of the map γ_f^*: Its classes therefore deserve to be called characteristic classes in the most classic sense that I have just explained.

7. Self-intersections of wave fronts. The use of generating functions to describe locally the legendre immersions gives analogous results in particular in the enumerative theory of the singularities of wave fronts. However, the most obvious enumerative result is obtained by none of these methods. Consider the big wave front of Figure 2 where we recognized a swallow tail (A_3). It suffices to consider the double points on this figure for an instant to be convinced that, if the legendre manifold is closed, it will necessarily have an even number of points of type A_3.

In order to obtain this result and others of the same kind, Vassilyev constructs in Chapter 3 some new complexes, this time taking into account several strata (not necessarily distinct) at a time to consider the intersections (such as the line of double points that we have just considered) of the points corresponding to these strata in the wave front. Here, the self-intersection of A_2 has $2 \cdot A_3$ as boundary. These complexes are naturally filtered (essentially by the number of strata considered), which permits us to calculate their cohomology thanks to a spectral sequence, and to deduce from it some results of the stated kind.

8. The book. The book is rather agreeable to read; I am thinking of the way the book has been made as well as its contents. It contains an introduction to lagrange manifolds and concludes with a list of open problems.

Its appearance is perhaps a bit late (undoubtedly because of the delays connected with the translation of the original text): Its results were obtained and published in two articles in the (Soviet) journal *Functional Analysis* in 1982. At this time, the calculation of lagrange and legendre cobordism groups was still in its infancy **[2]**, and the goal was to construct new invariants. After the homotopic description of Eliashberg **[6]** *via* the h-principle of Gromov, it was known how to compute these groups **[3]** by methods considered to be effective (algebraic topology). In my opinion, it is these results of "enumerative" type contained in the book that give this method its interest rather than the construction of *new* invariants, which are necessarily very limited because of the restriction on dimension that results from the method. These are obtained, to be sure at the price of incidence calculations, which are not always easy, by a *direct* method, while the derivation of the same results in [3] is given by more powerful but less direct methods.

However this may be, the style and the methods have the advantange of showing very concretely, and in an authentically muscovite ambience, the relations between the theory of singularities and a part of "symplectic topology," two specialities of the Arnold school.

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Unitary representations of reductive Lie groups, by David A. Vogan, Jr. Annals of Mathematics Studies, vol. 118, Princeton University Press, Princeton, N. J., 1987, x + 308 pp., $60.00 cloth, $19.50 paper. ISBN 0-691-08481-5, ISBN 0-691-08482-3

Group representations are the building blocks of harmonic analysis, a subject that dates historically from Fourier's use of superposition of sines and cosines in separating variables to study solutions of the heat equation. Fourier's theory generalizes in many directions; one of them is analysis of a space of complex-valued functions on a set on which a group acts.

A *group representation* is a homomorphism of the given group into invertible linear transformations on a complex vector space, usually topologized and usually with some continuity property in the group variable. If R is a group representation on the vector space V, we obtain some