the derivation of the same results in [3] is given by more powerful but less direct methods.

However this may be, the style and the methods have the advantange of showing very concretely, and in an authentically muscovite ambience, the relations between the theory of singularities and a part of "symplectic topology," two specialities of the Arnold school.

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Unitary representations of reductive Lie groups, by David A. Vogan, Jr. Annals of Mathematics Studies, vol. 118, Princeton University Press, Princeton, N. J., 1987, x + 308 pp., $60.00 cloth, $19.50 paper. ISBN 0-691-08481-5, ISBN 0-691-08482-3

Group representations are the building blocks of harmonic analysis, a subject that dates historically from Fourier's use of superposition of sines and cosines in separating variables to study solutions of the heat equation. Fourier's theory generalizes in many directions; one of them is analysis of a space of complex-valued functions on a set on which a group acts.

A *group representation* is a homomorphism of the given group into invertible linear transformations on a complex vector space, usually topologized and usually with some continuity property in the group variable. If R is a group representation on the vector space V, we obtain some complex-valued functions on the group as *matrix coefficients* $(R(g)u, v)$, where u is in V and v is in the dual of V. When R can be expressed as a sum or integral of representations, the matrix coefficients decompose in corresponding fashion. The ultimate reduction occurs if the constituent representations are *irreducible*, i.e., have no nontrivial invariant subspaces and in particular have no nontrivial invariant direct sum decompositions.

In one case of the kind considered by Fourier, the group in retrospect was the orthogonal group $O(2)$ in two dimensions. The solutions of the heat equation for the case in question are complex-valued functions of (x, t) with x in $[0, 2\pi]$ with the ends identified and with t in $[0, \infty)$. If we identify $[0, 2\pi]$ with the unit circle, then $O(2)$ acts on these functions by its effect in the x variable. The sines and cosines arise from matrix coefficients of irreducible representations, one for each integer $n \geq 0$. Fourier's method was to solve the equation by separation of variables. Equivalently, for each t, we decompose a solution $u(x, t)$ of the heat equation at t into its x constituents under the action of $O(2)$. When we use the equation, we obtain a restriction on the t dependence and are led to

$$u(x, t) = \sum_{n=0}^{\infty} e^{-n^2 t}(a_n \cos nx + b_n \sin nx).$$

The effect of the irreducibility of the nth representation of $O(2)$ is to constrain the corresponding t dependence as much as possible. In fact, the t dependence above was determined up to a constant factor as $e^{-n^2 t}$.

In the above case, the representations are *unitary* in the sense that they act in Hilbert spaces and the operators $R(g)$ are always unitary. Such representations have the property that the orthogonal complement of an invariant subspace is again an invariant subspace. This property facilitates the decomposition into irreducible constituents.

In fact, the techniques of harmonic analysis are severely limited in handling representations that are not unitary. For this reason, many of the successes of the method occur when the underlying set for the space of functions has a group-invariant measure; the space of functions may be taken to be the Hilbert space of square integrable functions, and then the natural representation of the group on this space of functions is unitary. Correspondingly, the interest in identifying and manipulating representations is greatest in the case of irreducible representations that are unitary.

To get concrete results from harmonic analysis, one needs concrete information about the irreducible unitary representations of particular groups. The initial theory for finite groups is due to Frobenius and Schur; the end theory for finite groups is not yet complete, not even for finite simple groups. For compact connected Lie groups, the irreducible unitary representations are fully understood. They are finite-dimensional and are parametrized as a consequence of the Cartan-Weyl theory of the 1920s. The centerpieces of this theory are the Theorem of the Highest Weight and the Weyl Character Formula. The celebrated Borel-Weil Theorem completes the theory for compact connected Lie groups by giving global realizations of the representations in terms of complex analysis.

For certain specific noncompact nonabelian connected Lie groups, the irreducible unitary representations were identified in the 1930s and 1940s by von Neumann, Wigner, Bargmann, and Gelfand-Naimark.

One can ask for what classes of particular groups it might be helpful to have a concrete parametrization of the irreducible unitary representations. Lie groups are a natural class. But there are too many Lie groups; finite groups are examples, and the representation theory of finite groups is far from completely understood. Connected Lie groups are a narrower natural class that eliminates finite groups from consideration, but for technical analytic reasons one wants to include some additional hypothesis. One such hypothesis is "type I." A stunning theorem of Duflo (cf. [**2**]), building on work of Mackey from the 1950s, reduces the parametrization of the irreducible unitary representations of all type I connected Lie groups to the case of Lie groups that are "semisimple."

A connected Lie group is *semisimple* if its Lie algebra is the direct sum of nonabelian ideals that contain no proper subideals. The groups $SL(n, R)$ and $SL(n, C)$ are examples. So is any other closed connected Lie group of real or complex matrices that is closed under conjugate transpose and has finite center. The most general example is an arbitrary covering group of such a group of matrices.

The irreducible unitary representations of connected semisimple Lie groups are the topic of the book under review. Despite great efforts, their classification remains an unsolved problem. A frustrating aspect of the theory is that one must go outside the class of groups under study in order to use arguments that induct on the dimension of the group. It is necessary to allow the group to be somewhat disconnected and to allow the Lie algebra to have abelian factors. There are several ways to enlarge the class of groups to allow an induction, and people differ on which one is best. In addition, there is disagreement whether the underlying connected semisimple group should be restricted to have finite center or further restricted so as to be a matrix group. Vogan makes a choice (different from everyone else's), and the result is his definition of "reductive Lie group."

So much for the book's title and the motivation for the study. The insides of the book divide neatly into two halves separated by an "interlude."

The first half discusses constructions of representations, particularly unitary representations. After an introductory overview, it begins with a long careful essay on compact Lie groups. One advantage of Vogan's definition of "reductive" is that all compact Lie groups are reductive. The chapter is beautifully written. Simultaneously it explains the traditional connected case and it shows the extent to which the theory in the disconnected case reduces to the theory for the finite group of components. The novice can learn the traditional theory here; the expert should pay attention to the subtle details of how the disconnectedness is handled.

The remaining chapters of the first half deal with constructions of representations in the noncompact case. A reductive group G in Vogan's sense has a maximal compact subgroup K. One important feature of the group G is that reducibility of representations of G can be understood on the level of the action of K and of the action of the Lie algebra of G. This theme derives from early work of Harish-Chandra, and it plays an increasingly important role as Vogan's book unfolds, beginning in Chapter 2.

After Vogan describes the real analysis concepts of parabolic induction and complementary series, he returns to the complex analytic theme begun in the discussion of the Borel-Weil Theorem. He describes Harish-Chandra's remarkable parametrization of "discrete series" representations and Schmid's global realization of these representations. The first half of the book ends triumphantly with the algebraic analog/generalization of this construction—cohomological parabolic induction—as introduced by Zuckerman [**13**] and elaborated by Vogan [**9**]. The final theorem is Vogan's own [**10**], describing how all the irreducible unitary representations of $GL(n, R)$ may be obtained by suitably combining the constructions of parabolic induction, cohomological parabolic induction, and complementary series, starting from the trivial representation.

The second half of the book is about unipotent representations. Vogan's hope is that the irreducible unitary representations for general G can be obtained by parabolic induction, cohomological parabolic induction, and complementary series constructions, starting from a finite set of representations he would like to characterize as "unipotent." As he says of unipotent representations, "All that they lack is a complete definition, a reasonable construction, a nice general proof of unitarity, and a good character theory." Five different tentative approaches to unipotent representations, some of them joint work with Dan Barbasch, occupy five chapters of the second half of the book. All of them are principally algebraic, no doubt betraying the author's personal taste. Yet the range of mathematics involved in these chapters is truly broad, extending from finite simple groups through algebraic number theory to noncommutative algebraic geometry.

In the field of representation theory of arbitrary noncompact reductive Lie groups, there are a few general books (Warner [**12**], Vogan [**9**], Knapp [**5**], and Wallach [**11**]) and a few mildly specialized books (Varadarajan [**8**], Borel-Wallach [**1**], and Knapp [**6**]). In addition, there are several books that develop aspects of representation theory in connection with the study of symmetric spaces (e.g., Helgason [**4**], Schlichtkrull [**7**], and Flensted-Jensen [**3**]). The present Vogan book fits into the category of mildly specialized books. But it is different in spirit from the rest, looking forward to unknown mathematics rather than merely organizing what is known.

The book gives real insight both into the status of a major unsolved problem and into the the thinking of a first-rate mathematician on solving the problem. Beginners and experts alike can profit from Vogan's ability to explain fine detail accurately without sacrificing a clear emphasis on the main ideas. The lack of an index is more than offset by extensive cross-referencing, incisive historical comments, and a lengthy bibliography. The book is printed on acid-free paper, as should be true of all mathematics books of any note.

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