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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 1, January 1990

0273-0979/90 $1.00 + $.25 per page

Asymptotic behavior of dissipative systems, by Jack K. Hale. Mathematical Surveys and Monographs, vol. 25, American Mathematical Society, Providence, R.I., 1988, ix + 198 pp., $54.00. ISBN 0-8218-1527-x

The dynamical systems encountered in physical or biological sciences can be grouped roughly into two classes: the conservative ones (including the Hamiltonian systems) and those exhibiting some type of dissipation. These dynamical systems are often generated by partial differential equations and thus the underlying state space is infinite dimensional.

I. It is natural to expect that the flow defined by a dissipative system shall be simpler than the one of a conservative system. It is perhaps even possible to isolate an interesting class of systems for which one can adapt several ideas coming from the ordinary differential equations (O.D.E's) to the analysis of the flow. If this can be done, then one must overcome the difficulties that arise due to the nonlocal compactness of the state space. This will require some type of "smoothing" property of the dynamical system. There are also problems that can arise at infinity due to the unboundedness of the space. This problem is avoided by imposing specific dissipative conditions. To make the discussion more meaningful and to motivate the class of systems considered in the book under review, it is instructive to recall the situation for the ordinary differential equations.

In his study of the forced van der Pol equation, Levinson [13] introduced the concept "point dissipative." To keep the technicality at a minimum, let us discuss at first discrete dynamical systems; that is, those defined by a map $T : \mathbb{R}^n \to \mathbb{R}^n$. The map T is *point dissipative* if there exists a bounded set B such that, for each $x \in \mathbb{R}^n$, there is an integer $n_0 = n_0(x, B)$ such that $T^n x \in B$ for $n \geq n_0$. If T is point dissipative, then due to the local compactness of $\mathbb{R}^n$, it is easy to show that T is *bounded dissipative* (equivalently, *uniformly ultimately bounded* or else there exists an *absorbing set*), that is, there is a bounded set B such that, for any bounded set U, one can find an integer $n_0(U, B)$ such that $T^n U \subset B$ for $n \geq n_0(U, B)$. For any bounded set B of $\mathbb{R}^n$, we define the ω-limit set $\omega(B)$ of B as $\omega(B) = \bigcap_{n\geq 0} \mathrm{Cl}(\bigcup_{j=1}^{n} T^j(B))$. If T is bounded dissipative, then the local compactness of $\mathbb{R}^n$ implies that $\omega(B)$ is compact and invariant for any bounded set B of $\mathbb{R}^n$ (i.e. $T\omega(B) = \omega(B)$). Therefore, if T is bounded dissipative and if we choose for B a ball of large enough radius, then $\omega(B) = \mathscr{A}$ is the *global attractor*; that is, $\mathscr{A}$ is compact, invariant and $\delta_{\mathbb{R}^n}(T^n U, \mathscr{A}) \to 0$ as $n \to +\infty$ for any bounded set U where $\delta_{\mathbb{R}^n}(T^n U, \mathscr{A}) = \sup_{x\in T^n U} \inf_{y\in\mathscr{A}} \|x - y\|_{\mathbb{R}^n}$. Note that this definition of the global attractor implies that $\mathscr{A}$ is maximal with respect to inclusion and hence unique. Thus, point dissipativness implies the existence of a global attractor (see Pliss [17]). This allows one to reduce the discussion to $\mathscr{A}$. Of course, if T is a mapping from M into M where M is a compact manifold without boundary, $\mathscr{A} = M$ (by uniqueness) and then nothing is gained.

Is it possible to have properties similar to the ones mentioned above for dynamical systems on $\mathbb{R}^n$ valid for an interesting class of dynamical systems on a Banach space X? To overcome the part of the difficulties at infinity, it is natural to require point dissipativeness as one of the fundamental properties. Unfortunately, due to the nonlocal compactness in infinite dimension, there are examples in which point dissipative does not imply that the orbits of bounded sets are bounded; where point dissipative and orbits of bounded sets are bounded does not imply bounded dissipative and where bounded dissipative does not imply that $\omega(B)$ is compact and invariant, if B is bounded. A counterexample to the latter implication can be constructed, for instance, for a wave equation, in which the damping $\beta(u)u_t$, with $\beta(u)$ continuous and bounded, satisfies $\beta(u) = 0$ for $|u| < r_0$ and $\beta(u) = \beta_0$ for $|u| > 2r_0$, where r_0 is a positive constant.

Thus, to have a theory comparable to the one for O.D.E's, one must impose a type of smoothing property on the operator T.

If some iterate of T is compact, Billotti and LaSalle [3] proved that point dissipative implies bounded dissipative which in turn implies the existence of a global attractor. Ladyzhenskaya [11, 12] has observed a similar result in the study of the Navier-Stokes equations on a bounded domain of $\mathbb{R}^2$.

If no iterate of T is compact, then the desired smoothing property is not clear. However it can be motivated in the following way. The orbits of bounded sets being bounded should be something that is observed about the system without reference to smoothing properties of T. Therefore we can ask the two following questions:

1. If the system is point dissipative and the orbits of bounded sets are bounded, which conditions on T are needed to have bounded dissipativeness?

2. What conditions on T do imply that the ω-limit set $\omega(B)$ is compact if the positive orbit $\gamma^+(B) = \bigcup_{n\geq 0} T^n B$ is bounded?

A local version of question 1 is related to a basic problem in stability theory of compact invariant sets. Let J be an invariant set; the set J is stable if, for any neighbourhood U of J, there is a neighbourhood V of J such that $T^n V \subset U$ for all $n \geq 0$. The set J attracts points locally if there is a neighbourhood W of J such that J attracts the points of W (We recall that J attracts a set C under T if, for any $\varepsilon > 0$, there is an integer $n_0 = n_0(\varepsilon, J, C)$ such that $T^n C$ belongs to the ε-neighbourhood $\mathscr{N}_X(J, \varepsilon)$ of J in X for $n \geq n_0$.) Finally the set J is *asymptotically stable* (a.s.) if J is stable and attracts points locally; the set J is *uniformly asymptotically stable* (u.a.s.) if J is stable and attracts a neighbourhood of J. If J is a compact invariant set, which conditions on T will imply that the properties of a.s. and u.a.s. for J are equivalent? A careful analysis of this question (see [7, Chapter 1]) leads naturally to the introduction of asymptotically smooth maps. A continuous mapping $T: X \to X$ is *asymptotically smooth,* if for any nonempty closed bounded set $B \subset X$, there is a nonempty compact set $J = J(B) \subset X$ such that $J(B)$ attracts the set $L(B) = \{x \in B; T^n x \in B, n \geq 0\}$.

One can prove the following results. If T is asymptotically smooth, point dissipative and if the orbit of any bounded set is bounded, then T is bounded dissipative. Furthermore, if T is asymptotically smooth and B is a nonempty bounded set such that the positive orbit $\gamma^+(B)$ is bounded, then $\omega(B)$ is nonempty, compact, invariant and attracts B. These properties lead us to the following existence theorem.

If T is asymptotically smooth, point dissipative and the orbits of bounded sets are bounded, there exists a connected global attractor $\mathscr{A}$.

Completely continuous maps are asymptotically smooth. Another example is given as follows: if, for each integer n, T^n is equal to $S_n + U_n$ where U_n is completely continuous and, for any $r > 0$, there is a constant $k(n, r)$ (with $k(n, r) \to 0$ as $n \to +\infty$) such that $\|S_n x\|_X \leq k(n, r)$ if $\|x\|_X \leq r$, then T is asymptotically smooth. More generally, conditional β-contractions are asymptotically smooth. We recall that a continuous map $T : X \to X$ is a conditional β-contraction of order k, $0 \leq k < 1$, with respect to the measure of noncompactness β if $\beta(TB) \leq k\beta(B)$ for any bounded set $B \subset X$ for which TB is bounded. The class of asymptotically smooth mappings contains many of the important dissipative dynamical systems encountered in the sciences. For instance, the time one map of the flow generated by certain damped wave equations is asymptotically smooth.

The above concepts of course apply equally well to the continuous dynamical systems, i.e., to the C^0-semigroups $T(t) : X \to X$, $t \geq 0$. Usually the backward initial value problem is not well posed for infinite dimensional C^0-semigroups. But, as the attractor $\mathscr{A}$ contains only complete orbits, the backward existence of solutions is ensured, so it remains to study the question of backward uniqueness of solutions on the attractor. For many partial differential equations (P.D.E.'s), this backward uniqueness on the attractor is true, while the class of functional differential equations (F.D.E's) for which backward uniqueness is valid is not well understood.

Although the above theorem is a powerful tool, in general, it is not obvious to show that an infinite dimensional dynamical system admits a global attractor. For instance, in the case of P.D.E's, one must prove at first that the equation generates a C^0-semigroup (a nontrivial fact in general). For several P.D.E.'s, one shows directly that the system is bounded dissipative by using a priori estimates (e.g. energy methods).

II. If a global attractor $\mathscr{A}$ exists, then any orbit will enter and remain in a small neighbourhood of $\mathscr{A}$ after a finite time. This does not mean that the transient behavior in a neighbourhood of $\mathscr{A}$ is unimportant. One expects that it will be reflected by the flow on $\mathscr{A}$. However this question has not received much attention in the literature. Of course, the flow on $\mathscr{A}$ can be extremely complicated because of the geometrical or topological structure of $\mathscr{A}$ as well as the dimension of $\mathscr{A}$ (Hausdorff dimension, fractal dimension or capacity).

The property of finite dimensionality of attractors was first proved by Mallet-Paret [14] for a general class of equations on a Hilbert space with an application to delay equations. Later, Foias, and Temam [5] proved the finite dimensionality of the attractor for the two-dimensional Navier-Stokes equations and also gave estimates of its dimension. There is a very general result due to Mañé [16]. If X is a Banach space, $T : X \to X$ is an α-contraction (where α is the Kuratowski measure of noncompactness), point dissipative and the orbits of bounded sets are bounded, then the capacity $c(\mathscr{A})$ is finite (since the Hausdorff dimension $d_H(\mathscr{A})$ is less or equal to $c(\mathscr{A})$, this implies that $d_H(\mathscr{A})$ is also finite). Moreover, if S is any linear subspace of X with $\dim S \geq 2\,d_H(\mathscr{A}) + 1$, there is a residual set Π of the space of all continuous projections P of X onto S such that $P/\mathscr{A}$ is one-to-one for P in Π. This means that $\mathscr{A}$ can be parametrized by at most $2\,d_H(\mathscr{A}) + 1$ parameters. In [8, Hale, Magalhães, and Oliva, Theorem 6.8], there is also a result of existence of retarded functional differential equations (R.F.D.E.'s) the attractors of which have infinite Hausdorff dimension.

If $\mathscr{A}$ has finite Hausdorff dimension, the result of Mañé says that $\mathscr{A}$ can be described by a finite number of parameters. From the point of view of understanding the flow on $\mathscr{A}$, it is natural to ask the following question: is the flow on the attractor $\mathscr{A}$ equivalent to the flow defined by a finite dimensional vector field? Even better, does the attractor belong to a positive invariant finite dimensional submanifold of the space? Such manifolds are called *inertial manifolds* and have been introduced in [4, 15], for reaction-diffusion equations primarily. Unfortunately, due to the techniques of construction of such manifolds (they use a gap property of the linear operator of the equation), up to now, only a few dissipative P.D.E.'s seem to have an inertial manifold. (The above topics are also mentioned in [6], for instance.)

For P.D.E.'s, there is almost no literature on the geometry of $\mathscr{A}$. For example, can $\mathscr{A}$ be a smooth manifold? For F.D.E.'s, there is some positive information about this question in [8]. There is a class of continuous dynamical systems for which one can describe the attractor in a more precise way; namely, the class of gradient systems. A gradient system on a Banach space X is defined to be a semigroup $T(t)$, $t \geq 0$, on X, for which there is a Lyapunov function. A continuous function $V : X \to \mathbb{R}$ is said to be a Lyapunov function for $T(t)$ if $V(\varphi)$ is bounded below, if $V(\varphi) \to +\infty$ as $\|\varphi\|_X \to +\infty$, $V(T(t)\varphi) \leq V(\varphi)$ for all $t \geq 0$, $\varphi \in X$ and if $V(T(t)\varphi) = V(\varphi)$ for all t implies that φ is an equilibrium point. If a gradient system has an attractor $\mathscr{A}$, then $\mathscr{A} = W^u(E) = \{x \in X;\, T(t)x$ is defined for $t < 0$ and $T(t)x \to E$ as $t \to -\infty\}$, where E is the set of equilibrium points. Moreover if each element of E is hyperbolic, E is finite of say N_0 points, and

$$\begin{aligned}\mathscr{A} &= \bigcup_{\varphi_j \in E,\, 1 \le j \le N_0} W^u(\varphi_j)\\ &= \bigcup_{1 \le j \le N_0} \{x \in X ;\, T(t)x \text{ is defined for } t < 0\\ &\qquad\qquad \text{and } T(t)x \to \varphi_j \text{ as } t \to -\infty\}.\end{aligned}$$

In numerous cases, the unstable sets $W^u(\varphi_j)$ are embedded submanifolds of X and one can give a Morse decomposition of $\mathscr{A}$. This is well described in [2, Babin, Vishik] as well as in the book under review.

One also can consider, for instance, a family of C^0-semigroups $T_\lambda(t)$ depending on a parameter $\lambda \to 0$ or, more generally, a family of perturbations $T_\lambda(t)$ which converge in some sense to $T_0(t)$. These perturbations can represent some variations of parameters in the equations, discretizations in space and time of P.D.E.'s etc... One can try to mimic what happens in finite dimension for O.D.E.'s and study the question of structural stability. Of course, due to the lack of local compactness in infinite dimension, we can expect to have stability results (with respect to perturbations) only on the attractors $\mathscr{A}_\lambda$. And one can ask how $\mathscr{A}_\lambda$ depends on λ. We say that $\mathscr{A}_\lambda$ is upper-semicontinuous at $\lambda = 0$ if $\delta_X(\mathscr{A}_\lambda, \mathscr{A}_0) \to 0$ as $\lambda \to 0$, where, for any two subsets A, B of X, $\delta_X(A, B) = sup_{x\in A} \inf_{y \in B} \|x - y\|_X$. We say that $\mathscr{A}_\lambda$ is lower-semicontinuous at $\lambda = 0$ if $\delta_X(\mathscr{A}_0, \mathscr{A}_\lambda) \to 0$ as $\lambda \to 0$ and that $\mathscr{A}_\lambda$ is continuous at $\lambda = 0$ if it is upper- and lower-semicontinuous at $\lambda = 0$. For instance, if the semigroup $T_\lambda(t)$ has the property that $T_\lambda(t)x$ is continuous in (t, x, λ), the continuity in λ being uniform with respect to t, x in bounded sets, then the attractors $\mathscr{A}_\lambda$ are upper-semicontinuous at $\lambda = 0$. Due to the strong stability properties of the attractors $\mathscr{A}_\lambda$, this upper-semicontinuity result remains true when the dependence of the semigroup in λ is worse. Even in the case of O.D.E.'s, lower-semicontinuity may not hold if $\lambda = 0$ is a bifurcation point. In the case of gradient systems, there exists a general result of lower-semicontinuity of $\mathscr{A}_\lambda$ at $\lambda = 0$, provided all the equilibrium points are hyperbolic [9].

By adapting the techniques used in finite dimension, it is proved in [8, Hale, Magalhães and Oliva] that if $T_0(t)$ has an attractor $\mathscr{A}_0$ and is Morse-Smale (that is, the nonwandering set is a finite set consisting only of hyperbolic equilibria and hyperbolic periodic orbits, with the stable and unstable manifolds transversal), if the time one maps $T_\lambda(1)$ converge to $T_0(1)$ in $C^1(U, X)$ as λ tends to 0, where U is a neighbourhood of $\mathscr{A}_0$ in X, then, under some additional assumptions, the systems $T_\lambda(t)$ are Morse-Smale and the corresponding flows restricted to the attractors $\mathscr{A}_\lambda$ are topologically equivalent for λ small enough. Now the natural question arises: are there interesting examples of Morse-Smale systems in infinite dimension? In 1985, Henry [10] (see also [1]) has proved the following nice result. For a scalar parabolic equation

$$u_t = u_{xx} + f(x, u, u_x)$$

with boundary conditions specified at $x = 0$ and 1, the stable and unstable manifolds of hyperbolic equilibria are always transversal. As one can choose the nonlinearity $f(x, u)$ so that the associated dynamical system is gradient and that the equilibria are all hyperbolic, this gives us a way to obtain Morse-Smale systems. In higher dimensions, no analogous result is known.

III. The book under review of Hale [7] and the recent book of Temam [18] are important contributions to the numerous publications that have appeared over the past twenty years on attractors and asymptotic behavior of PDE's, FDE's, etc... They are very different in spirit and the reader will profit in various ways from each.

In the three first chapters of his book, J. Hale explains which ideas arising from dynamical systems on locally compact spaces can be generalized to infinite dimensional systems and he develops the various concepts of stability and dissipation sketched in the first part of this review. He motivates in a very clear way the introduction of the family of asymptotically smooth mappings. By explaining the fundamental reasons for the existence of attractors, he clarifies why some equations or systems have an attractor, thereby eliminating some of the mystery that appears elsewhere. He also spends some time giving the general properties of the two important classes of systems, encountered before (whose attractors have special properties) namely the gradient and the Morse-Smale systems. In the last two thirds of the book, he gives many examples of retarded functional differential equations, neutral functional differential equations, and PDE's, indicating how to use the theory to get the existence of attractors. Although the abstract theory is powerful, one should not imagine that it can be applied in a trivial way and the author emphasizes that through his wide range of applications. For the scalar parabolic equations, Hale gives a detailed description of the structure of the attractors, including some results of bifurcation under perturbations. Finally he devotes a long interesting section to the semicontinuity and continuity of the attractors under regular as well as singular perturbations or under approximations, illustrating the results by several examples.

The book of R. Temam deals with the same topics giving a different presentation for the existence of attractors based on absorbing sets. His book concentrates on PDE's of physical interest, a major part being devoted to obtaining upper bounds on the dimensions of the attractors in terms of physical data (see [18, Chapter V]) and Lyapunov exponents. Considerable attention is devoted also to inertial manifolds. The presentation is very complete (e.g. even the needed existence and uniqueness results of solutions of the studied PDE's are recalled in an elegant way). This book and the book under review are complementary.

The aim of the author of the book under review was to present in detail the basic material on the subject of dissipative systems illustrated by advanced, recent applications and to try, through unsolved important problems, to bring other people to the study of this subject. In my opinion, this goal has been entirely reached; indeed, this monograph is a very good introduction to the subject and, at the same time, a place where the specialized reader can find many interesting open questions. Maybe one could regret that the author omitted some of the more complicated proofs in the applications, that he did not address the question of existence of inertial manifolds and estimates of the dimension of the attractors. However his set purpose of leaving out some subjects led to a gain in clarity.

This monograph is very pleasant to read and brings the reader in a short time through the fundamental ideas underlying the theory of infinite dimensional systems. In addition, the relevance of the theory for the applications is amply demonstrated.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 1, January 1990

0273-0979/90 $1.00 + $.25 per page

Real reductive groups. I, by Nolan R. Wallach. Academic Press, Pure and Applied Mathematics, San Diego, 1988, xix + 412 pp., $59.95. ISBN 0-12-732960-9

The subject matter of *Real reductive groups*.I is the harmonic analysis and representation theory of real reductive Lie groups. This book lays the groundwork for an eventual *Part* II which will