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Harmonic analysis on symmetric spaces and applications, by Audrey Terras. Springer-Verlag, Berlin, Heidelberg and New York, 1985, xii + 341 pp., $39.00 (paper), vol. 1. ISBN 0-387-96159-3 and 1988, x + 385 pp., $45.00 (paper), vol. 2 ISBN 0-387-96663-3

This two-volume set is a very informal, relatively elementary, and occasionally entertaining survey of parts of a highly-developed and very useful part of contemporary mathematics. Differential equations, special functions, number theory, physics, and statistics all make essential use of harmonic analysis (interpreted sufficiently broadly).

A prototype for a "symmetric space" is the unit circle $\mathbf{R}/\mathbf{Z}$; harmonic analysis is Fourier analysis. The fundamental idea is that *periodic functions can be "represented by" Fourier series.* Several things can be said about a Fourier series representation

$$f(x) \sim \sum_{n \in \mathbf{Z}} c_n e^{2\pi i n x}.$$

For f square integrable this is an equality in an L^2-sense, and

$$c_n = \langle f, \psi_n \rangle \qquad \langle f, f \rangle = \sum_n |c_n|^2 \qquad \left(\psi_n(x) = e^{2\pi i n x}\right).$$

Further, for another square-integrable function φ with Fourier series

$$\varphi(x) \sim \sum_{n \in \mathbf{Z}} d_n e^{2\pi i n x}$$

we have the *Parseval identity*

$$\langle f, \varphi \rangle = \sum_n c_n \overline{d}_n.$$

Pointwise convergence is more delicate; the series converges to f at points where f satisfies a Lipschitz condition, and convergence is absolute and uniform if f is smooth. If f is smooth, then the Fourier coefficients of f are rapidly decreasing.

Some things are so simple in this paradigmatic example that they may not be noticeable. First, the functions $x \to e^{2\pi i n x}$ (with $n \in \mathbf{Z}$) are the eigenfunctions for the one-dimensional Laplace operator $\Delta: f \to \partial^2 f / \partial x^2$. Integration by parts yields the self-adjointness of Δ; therefore, we might attempt to express $L^2(\mathbf{R}/\mathbf{Z})$

as a sum of eigenspaces for Δ. The expression of an L^2-function f by its Fourier series does this. Note that the *spectrum* of Δ, i.e., the set $\{-4\pi^2 n^2 : n \in \mathbf{Z}\}$ of eigenvalues for Δ, is a discrete subset of $\mathbf{R}$, and the dimension of each eigenspace is either 2 (for $\pm n$ with $n \neq 0$) or 1 (for $n = 0$). Second, the functions $x \to e^{2\pi i n x}$ are the continuous group homomorphisms from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{C}^x$. The Fourier series for a function expresses it as a linear combination of these representations. Third, the group $\mathbf{R}$ acts on functions on $\mathbf{R}/\mathbf{Z}$ by translations: $r \in \mathbf{R}$ gives an operator T_r defined by

$$T_r : (x \to f(x)) \to (x \to f(x+r)).$$

The exponential functions $\psi_n : x \to e^{2\pi i n x}$ are the eigenfunctions for these operators, with

$$T_r \psi_n = e^{2\pi i n r} \times \psi_n.$$

Further, translation-invariant differential operators on the circle further invariant under $x \to -x$ are polynomials in the Laplacian: the translation invariance requires that the operator be constant coefficient, and the invariance under $x \to -x$ eliminates odd-order terms.

Fourier series can also be used to understand *generalized functions* (distributions) on the circle $\mathbf{R}/\mathbf{Z}$. Consider the (periodic) "delta function" δ on periodic smooth functions f putatively defined by

$$\int_0^1 f(x)\delta(x)\,dx = f(0).$$

There is no such function; however, for smooth periodic functions

$$f(x) \sim \sum_n c_n e^{2\pi i n x}$$

we have

$$\int_0^1 \sum_n c_n e^{2\pi i n x} \delta(x)\,dx = \sum_n c_n.$$

From this, and from the Parseval identity, we might be tempted to write

$$\delta(x) \sim \sum_n e^{2\pi i n x}.$$

This series does not converge anywhere, and is not the Fourier series of an L^2 function. However, if we use a formula like Parseval's to evaluate the functional $f \to \delta(f)$ for a smooth function f, we get the right answer, so this series represents the periodic

delta functional in this sense. More generally, since the Fourier coefficients c_n of a smooth function f on $\mathbf{R}/\mathbf{Z}$ are rapidly decreasing, for any sequence $\{d_n\}$ of complex numbers growing polynomially in n, the sum $\sum_n c_n d_n$ is absolutely convergent. Therefore, the Fourier series $u \sim \sum_n d_n e^{2\pi inx}$ makes sense as a *distribution* i.e., as a functional on smooth functions. The usual rules for differentiating (nicely convergent) Fourier series apply to obtain the *distributional derivative* of a distribution written as a Fourier series: e.g., the n th derivative of the periodic delta is

$$\delta^{(k)} \sim \sum_n (2\pi in)^k e^{2\pi inx}.$$

A subtler elementary example is Fourier analysis on $\mathbf{R}$. Define the *Fourier transform* $\hat{f}$ and the inverse transform $\check{f}$ of an integrable function f on $\mathbf{R}$ by

$$\hat{f}(x) = \int_{\mathbf{R}} f(u) e^{2\pi ixu}\, du \qquad \check{f}(x) = \int_{\mathbf{R}} f(u) e^{2\pi ixu}\, du.$$

For suitable functions f, we recover f from $\hat{f}$ by the *Fourier inversion formula*

$$f(x) = \hat{f}\check{}\,(x) = \int_{\mathbf{R}} \hat{f}(u) e^{2\pi ixu}\, du.$$

For example, this holds for smooth f so that f and all its derivatives are rapidly decreasing at infinity (Schwartz functions). There is a *Parseval identity*

$$\langle f_1, f_2 \rangle = \langle \hat{f}_1, \hat{f}_2 \rangle$$

where

$$\langle f_1, f_2 \rangle = \int_{\mathbf{R}} f_1(x) \overline{f}_2(x)\, dx.$$

Here, by contrast with the Fourier series situation, the inversion formula represents functions as *integrals* of exponential functions, rather than *sums*, and the exponential functions are no longer themselves square-integrable. Still, integration by parts shows that

$$(\partial f/\partial x)\hat{} = 2\pi ix \times \hat{f}$$

so that Fourier transforms turn differentiation into multiplication, and the exponentials are eigenfunctions for the Laplacian $\Delta = \partial^2/\partial x^2$. Similarly, letting $\mathbf{R}$ act on $L^2(\mathbf{R})$ by translation operators, we find that there are no eigenfunctions (obviously), and that the spectral decomposition given by the Fourier inversion formula is an *integral* of exponentials, which *are* eigenfunctions for these translation operators.

The theory of Fourier series on $\mathbf{R}/\mathbf{Z}$ yields an identity between certain tempered distributions on $\mathbf{R}$, the *Poisson summation formula*: let f be a Schwartz function on $\mathbf{R}$; then

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{n\in\mathbf{Z}} \hat{f}(n).$$

To prove this, define a smooth function F on $\mathbf{R}/\mathbf{Z}$ by

$$F(x) = \sum_{n\in\mathbf{Z}} f(x+n).$$

Equating $F(0)$ with the Fourier series of F evaluated at 0 gives the Poisson summation formula directly. Among many interesting identities derivable from the Poisson summation formula there is *Jacobi's theta identity*

$$\sum_{n\in\mathbf{Z}} \exp(-\pi n^2 y) = y^{-1/2} \sum_{n\in\mathbf{Z}} \exp(-\pi n^2/y) \qquad (y > 0).$$

Here we play upon the possibility of explicitly computing a Fourier transform, and also that the Fourier transform of $x \to \exp(-\pi x^2 y)$ is of a similar sort. From Jacobi's theta identity Riemann obtained the *analytic continuation*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = -1/s - 1/(1-s) + \text{ entire}$$

and *functional equation*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

of the zeta function

$$\zeta(s) = \sum_{n\geq 1} 1/n^s$$

via an integral transform trick. From this, Hadamard and de la Vallee-Poussin proved the *Prime Number Theorem*

$$\text{number of primes less than } x \sim x/\log x.$$

Now consider the unit $(n-1)$-sphere S^{n-1} in $\mathbf{R}^n$, with the linear action of the orthogonal group $SO(n)$. For $n = 2$ we just have the circle $\mathbf{R}/\mathbf{Z}$ and the orthogonal group $SO(2)$ is also a circle, so nothing new is happening. Having better judgement than to try to write spherical coordinates in n dimensions, we might define a "Laplacian" Δ on the sphere as follows. Given a smooth function f on the sphere, define a smooth function $\tilde{f}$ on $\mathbf{R}^n - 0$ by

$$\tilde{f}(x) = f(x/\|x\|).$$

Then for $x \in S^{n-1}$ put

$$\Delta f(x) = \sum_i (\partial^2/\partial x_i^2)\tilde{f}(x) \text{ restricted to } S^{n-1}.$$

It is not so hard to check that this differential operator is rotation-invariant, and is selfadjoint with respect to integration against a rotation-invariant measure on the sphere. In fact, any rotation invariant differential operator is provably a polynomial in this Δ. By Weierstrass approximation, the space V of functions on S^{n-1} given by restrictions of polynomials on $\mathbf{R}^n$ is dense in $L^2(S^{n-1})$. Some simple considerations also show that the space

$$\mathscr{H} = \left\{ \text{polynomials on } \mathbf{R}^n \text{ annihilated by } \sum_i \frac{\partial^2}{\partial x_i^2} \right\}$$

map surjectively to V under restriction to the sphere. Let $\mathscr{H}(d)$ be the finite-dimensional space of homogeneous degree d elements of $\mathscr{H}$. Then we readily compute that Δ acts on $\mathscr{H}(d)$ by the scalar $-d(n+d-2)$. Therefore, *the spectrum of* Δ *on the sphere is discrete, with finite multiplicities.* The restrictions of elements of $\mathscr{H}$ to the sphere are *spherical harmonics*, and for distinct degrees the spherical harmonics are orthogonal, because the eigenvalues are distinct. The theory of *distributions on the sphere* is explicable in terms of expansions in spherical harmonics, in analogy with distributions on $\mathbf{R}/\mathbf{Z}$.

Generalizing $\mathbf{R}/\mathbf{Z}$, Euclidean spaces, and spheres, *symmetric spaces* are connected Riemannian manifolds M such that for each point $x_0 \in M$ there is a geodesic-reversing isometry fixing x_0. A symmetric space can be factored into irreducible factors, each of which is a symmetric space. Irreducible symmetric spaces are of three types: Euclidean ($\approx \mathbf{R}$), compact, and noncompact. The Euclidean case is very familiar, and the compact symmetric spaces are all quotients of compact simple Lie groups by closed subgroups, so are relatively easy to understand. The irreducible noncompact symmetric spaces are all quotients G/K where G is a noncompact simple real Lie group and K is a maximal compact subgroup of G. Thus, the isometry group G of a symmetric space M acts transitively on M. E. Cartan showed that every symmetric space with no Euclidean factor is a quotient G/K for a semisimple Lie

group G and a compact subgroup K. Since G has a two-sided Haar measure, there is a left G-invariant measure on G/K and a right G-invariant measure on quotients $\Gamma \backslash G$ of G by discrete subgroups Γ. Functions on G/K obviously can be turned into right K-invariant functions on G and vice-versa, and functions on $\Gamma \backslash G/K$ are left Γ-invariant right K-invariant functions on G, which are right K-invariant functions on $\Gamma \backslash G$, etc. The point is that we can do analysis on $\Gamma \backslash G$ without loss of generality, a fundamental object being the space $L^2(\Gamma \backslash G)$ of square-integrable functions. Note that $\Gamma \backslash G/K$ generally does not have an action of G upon it, so we cannot discuss G-invariant objects thereupon; the change from analysis on G/K to G itself, as obvious as it now may seem, was no small step historically. (The structure of Riemannian manifold of G/K should not be entirely forgotten.) The book [H] provides a systematic treatment of analysis on symmetric spaces.

The algebra of G-invariant differential operators on G/K is a homomorphic image of the commutative finitely-generated algebra Z of left and right G-invariant differential operators on G itself. The operators in Z naturally descend to right G-invariant operators on quotients $\Gamma \backslash G$. A fundamental problem is *spectral decomposition of* $L^2(\Gamma \backslash G)$ *with respect to* Z. G acts on $L^2(\Gamma \backslash G)$ by right translation π, giving unitary operators, and we can ask to *decompose* $L^2(\Gamma \backslash G)$ *with respect to this action* π *of* G (into irreducible unitary representations of G). Since $\pi(G)$ commutes with the operators in Z, Schur's lemma shows that these two spectral decomposition problems are intimately related. If we only look at $L^2(\Gamma \backslash G/K)$ then there are no representations of G in evidence, since G generally does not act on the double quotient $\Gamma \backslash G/K$.

Having more or less converted the problem of decomposition with respect to differential operators into a decomposition problem regarding group representations, we could drop the hypothesis that G be a Lie group, and pose the question more generally: *for a unimodular topological group* G *and a closed unimodular subgroup* Γ *decompose* $L^2(\Gamma \backslash G)$ *into irreducible representations of* G.

When $\Gamma \backslash G$ is compact (e.g., when G itself is compact), $L^2(\Gamma \backslash G)$ decomposes *directly with finite multiplicities* into irreducible representations of G; therefore, in the Lie group case, Schur's lemma shows that $L^2(\Gamma \backslash G)$ has a Hilbert space basis of eigenfunctions for the differential operators in Z. When $\Gamma \backslash G$ is noncompact, there is typically some continuous part in the spectral decomposition, as well.

Let us prove the discreteness in the case of compact $\Gamma \backslash G$. For a compactly-supported continuous function φ on G, we have a linear operator on $L^2(\Gamma \backslash G)$ given by

$$\pi_i(\varphi)f(g) = \int_G \varphi(h)f(gh)\,dh.$$

This can be rearranged as

$$\pi_i(\varphi)f(g) = \int_{\Gamma\backslash G} f(h)\left[\sum_\Gamma \varphi(g^{-1}\gamma h)\right] dh.$$

The kernel

$$K_\varphi(g, h) = \sum_\Gamma \varphi(g^{-1}\gamma h)$$

is readily seen to be continuous on $\Gamma \backslash G \times \Gamma \backslash G$, *so, by compactness is square-integrable.* Therefore, $\pi_i(\varphi)$ is a *compact operator.* From the most elementary properties of compact operators, we find that $L^2(\Gamma \backslash G)$ decomposes *discretely with finite multiplicities* as a sum of irreducible representation spaces for the algebra of operators $\pi_1(\varphi)$. From this (and from the existence of approximate identities) we conclude that $L^2(\Gamma \backslash G)$ decomposes *discretely with finite multiplicities* as a sum of irreducible representation spaces for G.

A problem not merely incidental to the spectral decomposition of $L^2(\Gamma \backslash G)$ is the *classification of the irreducible unitary representations* of G. For abelian groups this is an easy problem in the abstract: this is the theory of Fourier series and Fourier transforms extended only modestly. For compact groups all irreducible unitary representations are finite-dimensional, and for compact Lie groups are well described in terms of highest weights. In general, one hopes for a description of such representations as subrepresentations of representations induced from "simple" representations of closed subgroups. The notion of induced representation is technically important, and is defined as follows for G and Γ both unimodular. Let $\rho : \Gamma \to U(H)$ be a unitary representation of Γ on a Hilbert space H with inner product $(\ ,\ )$. Let V_0 be the space of continuous H-valued functions f on G which are compactly supported modulo Γ and so that

$$f(\gamma^{-1}g) = \rho(\gamma)f(g)$$

for all $\gamma \in \Gamma$ and $g \in G$. Give V_0 an inner product

$$\langle f, \varphi\rangle = \int_{\Gamma\backslash G} (f(g), \varphi(g))\,dg.$$

Then the *induced representation of* ρ *from* Γ *to* G is the representation of G on the completion of V_0 with respect to $\langle\ ,\ \rangle$. This immediately suggests a still more general version of the problem: *given a representation of a closed subgroup* Γ *of a group* G, *decompose the induced representation on* G *into irreducible representations of* G. When G is compact, a complete answer to this question is provided by *Frobenius reciprocity*: the multiplicity with which an irreducible representation π occurs in the induced representation of ρ is equal to the multiplicity with which ρ occurs in the restriction of π to Γ.

The simplest example of classification of irreducible unitary representations in the noncompact and nonabelian case is $SL(2, \mathbf{R})$. Apart from the trivial representation, there are two families of representations, essentially described as follows (see [GG, K]). The *continuous series* of representations consists of representations induced from one-dimensional representations

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \to |a|^s \operatorname{sign}(a)^\delta$$

of the (parabolic) subgroup of upper triangular matrices in $SL(2, \mathbf{R})$; the *discrete series* of representations consists of representations induced from the representation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \to e^{ik\theta}, \qquad (|k| \geq 2)$$

of the circle group $SO(2)$.

The spectral decomposition of $L^2(\Gamma \setminus G)$ with $\Gamma = SL(2, \mathbf{Z})$ and $G = SL(2, \mathbf{R})$ is intimately connected with the theory of *modular forms*. Let U be the group of upper triangle unipotent elements of G; the space of *cuspforms* is defined to be

$$\left\{ f \in L^2(\Gamma \setminus G) : \int_{U \cap \Gamma \setminus U} f(ug)\, du = 0 \text{ for almost all } g \in G \right\}.$$

It is a fundamental result that the discrete part of $L^2(\Gamma \setminus G)$ consists of (constants and) *cuspforms*. (See [G1, G2, GG].) A discrete series representation with $k \geq 2$ occurring in $L^2(\Gamma \setminus G)$ is generated by (derivatives of) a single function $\tilde{f}$ which is a *holomorphic cuspform* of weight k for Γ "transplanted" from the complex upper half-plane to G. A continuous series representation occurring discretely in $L^2(\Gamma \setminus G)$ is generated by (derivatives of) a single function $\tilde{f}$ which is a *Maass waveform* for Γ transplanted from the complex upper half-plane. Therefore, we obtain a kind of

reciprocity law: the multiplicity with which an irreducible representation of G occurs in $L^2(\Gamma \setminus G)$ is the dimension of the space of automorphic forms of the corresponding type.

There is a curious relationship between modular forms and the *spectra of (flat) tori*. Consider a torus $M = \mathbf{R}^n/\Lambda$ with the Laplacian coming from $\mathbf{R}^n$, where Λ is some lattice in $\mathbf{R}^n$. Let Q be a nonsingular matrix so that $Q^T\Lambda = \mathbf{Z}^n$. The theory of Fourier series in several variables shows that eigenfunctions for Δ on M are given by the exponential functions

$$x \to \exp(2\pi i\langle x, Q\xi\rangle), \quad (\xi \in \mathbf{Z}^n).$$

Therefore, the spectrum of the Laplacian on M is the set of values

$$\{\langle Q\xi, Q\xi\rangle : \xi \in \mathbf{Z}^n\}.$$

The *theta series*

$$\theta(z) = \sum_{\xi \in \mathbf{Z}^n} \exp(\pi i\langle Q\xi, Q\xi\rangle z)$$

which is the generating function for the values $\langle Q\xi, Q\xi\rangle$ (with $\xi \in \mathbf{Z}^n$) can be shown to be a *modular form* of weight $n/2$ for some subgroup of $SL(2, \mathbf{Z})$ of finite index. This fact, together with some elementary properties of holomorphic modular forms, yields some rather startling *arithmetic* features of the spectrum of these tori, such as

number of ways to write n as sum of 8 squares =

(= multiplicity of $-\pi^2 n$ in spectrum of $\mathbf{R}^8/\mathbf{Z}^8$ =)

$= 16 \times$ sum of cubes of divisors of n (for positive odd n).

A result which bears upon computation of multiplicities and reciprocity laws (among many other things) is the *trace formula*. Let G be a unimodular group and Γ a discrete subgroup. We let G act on $L^2(\Gamma \setminus G)$ by right translation π, and let π_1 be the representation of continuous compactly-supported functions on G on the Hilbert space $L^2(\Gamma \setminus G)$ by

$$\pi_1(\varphi)(f) = \int_G \pi(h)(f)\varphi(h)\,dh.$$

Suppose that $\Gamma \setminus G$ is compact. We saw above that the map $f \to \pi_1(\varphi)f$ is given by integration against a square-integrable kernel $K_\varphi(g, h)$: $\pi_1(\varphi)$ is a *compact operator*. If the notion of *trace* were to make sense, then by general integration theory

$$\operatorname{trace} \pi_1(\varphi) = \int_{\Gamma\setminus G} K(h, h)\,dh = \sum_{\{\alpha\}} \int_{\Gamma_\alpha \setminus G} \varphi(h^{-1}\alpha h)\,d\gamma$$

where $\{\alpha\}$ is the conjugacy class of α and Γ_α is the centralizer of α in Γ. Letting G_α be the centralizer of α in G, this becomes

$$\operatorname{trace} \pi_1(\varphi) = \sum_{\{\alpha\}} \left\{ \int_{G_\alpha \backslash G} \varphi(h^{-1}\alpha h)\, d\gamma \times \operatorname{volume}(\Gamma_\alpha \backslash G_\alpha) \right\}.$$

Because $\pi_1(\varphi)$ is a *compact operator* $L^2(\Gamma \backslash G)$ decomposes *discretely* as

$$L^2(\Gamma \backslash G) = \widehat{\bigoplus} \mu_\beta V_\beta$$

where β runs over irreducible unitary Hilbert space representations of G and the μ_β's are integers. Therefore,

$$\begin{aligned} \sum_\beta \mu_\beta \operatorname{trace} \beta_1(\varphi) &= \operatorname{trace} \pi_1(\varphi) \\ &= \sum_{\{\alpha\}} \left\{ \int_{G_\alpha \backslash G} \varphi(h^{-1}\alpha h)\, d\gamma \times \operatorname{volume}(\Gamma_\alpha \backslash G_\alpha) \right\}. \end{aligned}$$

This is the *trace formula* (in the compact quotient case), which should be interpreted as an equality of *distributions* on G. (See [S, He, GG].)

In the case that G is *abelian* (e.g., for $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$), the *trace formula* simplifies to

$$\sum_\beta \mu_\beta \operatorname{trace} \beta_1(\varphi) = \operatorname{trace} \pi_1(\varphi) = \operatorname{volume}(\Gamma \backslash G) \sum_{\alpha \in \Gamma} \varphi(\alpha).$$

All the irreducible unitary representations of abelian groups are 1-dimensional, so $\operatorname{trace} \beta_1 = \beta_1$;

$$\sum_\beta \mu_\beta \beta_1(\varphi) = \operatorname{trace} \pi_1(\varphi) = \operatorname{volume}(\Gamma \backslash G) \sum_{\alpha \in \Gamma} \varphi(\alpha).$$

Writing out the definition of $\beta_1(\varphi)$, this is

$$\sum_\beta \mu_\beta \int_{\Gamma \backslash G} \beta(g)\varphi(g)\, dg = \operatorname{trace} \pi_1(\varphi) = \operatorname{volume}(\Gamma \backslash G) \sum_{\alpha \in \Gamma} \varphi(\alpha).$$

Using a notation suggested by Fourier series on $\mathbf{R}/\mathbf{Z}$, we could write

$$\sum_\beta \mu_\beta \hat{\varphi}(\beta) = \operatorname{volume}(\Gamma \backslash G) \times \sum_{\alpha \in \Gamma} \varphi(\alpha).$$

For $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$ this is nothing but the *Poisson summation formula.* It is for this reason that the trace formula is often termed a *nonabelian Poisson summation formula.*

For reasons sketched above, much of the harmonic analysis on symmetric spaces has been converted into the study of representations of Lie groups, to great technical advantage. The latter is highly-developed: see [K] for a serious introduction. Further, the harmonic analysis on quotients $\Gamma \backslash G$ of Lie groups G by arithmetically-defined discrete subgroups Γ is likewise advantageously reconsidered as harmonic analysis on quotients of adelized groups, whereupon arithmetic objects (such as Hecke operators) have the same status as analytical objects (such as differential operators). (See [GG and JL].)

In Terras' two volumes there are some proofs or sketches of proofs, and a voluminous though selective bibliography. To many mathematicians the extra-mathematical applications and references may be surprising, or at least amusing. Many important and interesting results and applications are mentioned, but many ideas introduced are not carried through to completion; this may frustrate some readers. Of course, these volumes would have to be vastly larger if even a fraction of the topics mentioned were treated in detail: the selective neglect of details allows discussion of a greater number of topics, but also necessitates a corresponding superficiality at many points. The first volume, treating the simplest examples (e.g., $\mathbf{R}^n$, $\mathbf{R}^n/\mathbf{Z}^n$, S^2, and $SL(2, \mathbf{R})$) gives the most complete treatment of its subject matter.

These two volumes are an engaging introduction for students and nonspecialists not too interested in hard details beyond a certain point. The writing style is very informal and "friendly;" this, together with the modest demands upon the mathematical experience of the reader, create a pleasant impression of accessibility. The bibliography is sufficient to provide an initial introduction to the literature.

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Etale cohomology and the Weil conjecture by Eberhard Freitag and Reinhardt Kiehl. Springer-Verlag, Berlin, Heidelberg, New York, 1988, xviii + 317 pp., $98.00. ISBN 3-540-12175-7

It is now some sixteen years since Deligne's spectacular proof [De-Weil I] in June, 1973, of the "Riemann Hypothesis" for zeta functions of projective nonsingular varieties over finite fields completed the overall proof of the Weil Conjectures [We]. For an expository account of all this, see my survey article [Ka].

In the fall of 1973, Deligne formulated and proved [De-Weil II] a far-reaching generalization, which applied to arbitrary varieties over finite fields, and to quite general L-functions on them. It is this generalization, rather than Weil I itself, which has since proven an extremely powerful tool with all sorts of applications, from exponential sums to perverse sheaves.

The book under review is devoted to giving a thorough exposition of Weil I, and of the background material concerning Grothendieck's theory of l-adic cohomology which that paper presupposes. In this the authors succeed admirably. The book does not discuss Weil II at all, except for a two page summary (IV, 5) of some of its main results near the end. Perhaps someday if the authors feel ambitious... .

The excellent 1975 survey article of Dieudonne [Di] on the Weil Conjectures and their solution has been reprinted in the present book as an "historical introduction." Thus the reader has no problem in knowing from the beginning what the "point" of the book is. And if he keeps open a copy of Weil I, which is only 34 pages long, he will not lose his way as he reads through the book. The quality of the exposition is quite high, although the book is (necessarily, being of finite length) not self-contained, and occasionally anachronistic. For instance, on pp. 63–64 Artin approximation (1969) is