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Lest the title leave any uncertainty, Volume 1 initiates a comprehensive four-part overview of universal algebra as the subject is understood today. It concerns properties of *algebras* that are, by and large, independent of their particular operational type. Special algebras, such as *lattices*, are dealt with from the point of view of their role in the study of universal algebras (nonempty sets augmented with an arbitrary system of finitary operations). *Varieties*, or equational classes of algebras, arise as one of the central themes in universal algebra. This volume presents a thorough and exquisitely executed account of the foundations of universal algebra together with a fine exposition of several sample results that illustrate the depth and the beauty of the subject.

The sheer quantity of new work published in universal algebra makes a strong case for the need for such a series. The Mathematical Reviews' Mathematics Subject Classification encompasses most of the universal algebra in two categories: 06XXX Order, Lattices, Ordered Algebraic Structures; and 08XXX General Mathematical Systems. But the 1970 version, which offered the single letter sections 06AXX and 08AXX, quickly proved to be a poor reflection of the explosion of research that was erupting. Grätzer [3] estimated that about a thousand publications in universal algebra appeared between 1968 and 1979, and it seems likely that an equal
number have appeared since. Responding to such areas of increased activity, the 1980 Classification significantly reorganized and expanded ten of its sixty categories. Among these ten were both 06XXX, expanded to 06AXX, 06BXX, 06CXX, 06DXX, 06EXX, and 06FXX; and 08XXX, expanded to 08AXX, 08BXX and 08CXX.

So what is the impetus for all of this work? Algebra, as a tool for the study of mathematical structures apart from the classical number systems, was originated and established in the nineteenth century. Groups, rings, fields, Boolean algebras, vector spaces, semigroups, and lattices arose as useful abstractions to model a spectrum of mathematical objects. More focused study of the properties and structure of each of these classes led to a deeper understanding of the objects they modeled. Much of this work was assembled in A. N. Whitehead’s 1898 *A Treatise on Universal Algebra* [4] where, for the first time, a unified approach to these varied algebras was proposed. Many fundamental notions proved to be totally independent of the choice of a particular operational type of algebras: subalgebras, quotients, direct products, and automorphisms, as a beginning. Likewise, the proofs of many results from special algebra were shown to rely on hypotheses expressible in terms of only these notions. The experience of the past sixty years has repeatedly shown that many new and deep results about algebras depend on conditions which have nothing to do with a special operational type. A new dimension of insight arises when we turn from the historical classifications of algebras as groups, modules, monoids, etc., to investigate instead algebras generating congruence distributive, modular or permutable varieties, locally finite algebras, algebras with decidable first-order theories, with solvable word problems or perhaps with very large clones. This fact is the central driving force behind universal algebra.

As the body of knowledge in this field has grown in breadth and depth, the task of digesting it into textbook form has become increasingly formidable. Perhaps G. Grätzer took the last opportunity to become a renaissance man of universal algebra when, in 1968, he published a comprehensive account of the knowledge of the day [2]. His second edition [3], appearing in 1979, made no attempt to repeat this feat. Rather, it stood on its reputation as a sound introduction to the foundations of the discipline and added, as separate appendices by different authors, accounts of
subsequent work in five major areas. In 1981 the book by Burris and Sankappanavar [1] appeared. This relatively compact, well-composed, and very successful text again made no pretense of being exhaustive. It gave a thorough review of basic principles and then efficiently led the reader to an exposition of several chosen highlights, particularly those at the junction of universal algebra and mathematical logic. But, in the meantime, the absence of a reasonably comprehensive account of the last twenty years' developments has left for the novice insufficient access to a reservoir of folklore which is viewed as well known by the experts.

It is this latter need that the present series promises to fill, and Volume 1 is a major step toward doing so. The exposition is clear, clean, and very readable. Frequent references, both forward and backward, help the reader orient the material in a broader context. Citations to earlier (but easily forgotten) definitions are often provided. References to relevant historical motivation are tightly interwoven with the development of mathematical content. Many sections end with references to either (an apparently well-formulated) Volume 2 or (more vaguely) to "subsequent volumes." Problems and exercises at the end of each section are ample and substantive.

The topics presented in this text seem to fall into three categories: development of the standard tools used to understand algebra, presentation of examples to which these tools can be applied, and extraction of major results which demonstrate the power and strength of these tools. I will address each of these categories separately.

Chapters 1, 2, and 4, which form the bulk of the book, are primarily dedicated to a systematic development of the fundamental concepts and working tools of universal algebra. Chapter 1 provides the notational and conceptual framework required for subsequent work.

Chapter 2 gives both a logical and historical account of the central and ubiquitous role of lattices in the study of algebra. Here we see how the emergence of lattices of congruences, subalgebras, clones, and varieties in universal algebra leads to a concentrated study of lattice theory itself. Many classical results, such as the Jordan–Hölder theorem, are seen to arise from the modularity of the underlying congruence lattices, thus having led Dedekind, Ore, and Birkhoff to initiate a careful study of modular lattices. Simi-
larly it is the distributivity of their congruence lattices, apart from any other properties associated with their operational type, that accounts for many of the special properties of lattices. (For example, Baker's proof that a finite algebra in a congruence distributive variety has a finite equational basis for its identities was suggested by McKenzie's earlier proof for the special case of lattices.) Distributive lattices manifest very strong properties since they turn out to be exactly the lattices representable by set union and intersection.

Chapter 4, the real core of the text, utilizes the previously established foundation to give a smooth and coherent exposition of a large number of major fundamental results. Clones are carefully treated for future use since the universal algebraist is often not concerned with any particular choice of generating operations. Next the authors treat several topics which were first done in the context of various special algebras, but which rely in no way on their special properties: the isomorphism theorems, congruence generation, and subdirect and direct representation. Since most lattices that arise from algebras are algebraic lattices, this class of lattices also earns a systematic treatment. Congruence distributivity and \((n-)\)permutability are known to be responsible for many of the important properties that have been found in special algebras. Since these topics are presented before the section on free algebras, the otherwise easy proofs of their Mal'tsev characterizations are not yet accessible. But this problem is remedied when, later, they find their places in the section on interpretation of varieties. Class operators, free algebras, and Birkhoff's theorem come next. Chapter 4 concludes with an introduction to the commutator, focusing on Abelian algebras, and a promise to treat this important tool in full in Volume 3.

Examples and illustrations of these concepts occur throughout the book in the form of numbered examples and exercises. They demonstrate the richness of applicability of the ideas and provide short introductions to a wide selection of subsidiary topics. Chapter 1 and (most of) Chapter 3 give further concrete examples from which to draw later intuitions. As we see from Chapter 3, most of the complexity of the subject can already be found in binary algebras.

Finally, the authors have not failed to exposit a selection of major results built with the tools they have developed. These results are sprinkled through Chapters 2 and 4 in the form of several
classical decomposition and representation theorems for distribu­
tive lattices, modular lattices, and projective geometries. But the
primary harvest of terminal results occurs in the fifth chapter on
unique factorizations of direct products into indecomposable fac­
tors. Here the authors have chosen a special advanced topic which
has historically drawn people to universal algebra and lattice the­
ory, and which itself builds on many components of the previous
work. Unique factorization of finite sets (positive integers) into
primes dates back to the early Greeks. Classical algebra abounds
with unique factorization theorems: finite Abelian groups (Kro­
necker); groups, rings, and modules with chain conditions (Re­
mak, Krull, Schmidt); finite-dimensional linear spaces (Wedder­
burn); projective geometries of finite height (Jónsson); and finite
Boolean algebras. The fascination of this subject arises from sev­
eral startling theorems which assure unique factorization under
surprisingly weak hypotheses. Their proofs typically rely on our
ability to recognize a decomposition by its projection congruences
in the congruence lattice. Examples of such weak hypotheses are
having permuting congruences, a finite congruence lattice, and a
one-element subuniverse (Birkhoff, Ore); having modular congru­
ences and being finite with a one-element subuniverse (Jónsson),
and being finite with a one-element subuniverse that is a two-sided
identity under some binary operation (Jónsson and Tarski). An
unexpected result with a deceptively easy but most unconventional
proof is that every finite algebra has at most one $k$th root for ev­
every $k$ (Lovász). Weak as these hypotheses are, the authors exhibit
a multitude of examples to show how unique factorization fails
under almost any relaxation of any of them.

The organization and choice of topics lend the book to a selec­
tion of different uses: an introductory text for the beginner, an
advanced text for the intermediate, or a reference book for the
expert. A good beginning graduate course might cover Chapters 1
and 2, and then include topics selected from Chapter 3 or later
chapters. For a more advanced course, or for the reader having
some working familiarity with the subject, a quick review of the
first part of the book would be sufficient to begin Chapter 4. The
fifth chapter offers an excellent opportunity to pursue one possible
avenue into more advanced topics.

Finally, I would say that Volume 1 is a timely, important, and
well-constructed contribution to mathematics, and I would highly
recommend it to anyone at all interested in universal algebra. For my own part, I will be anxiously awaiting the appearance of Volumes 2, 3, and 4.

REFERENCES


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In the discussion on pages 255–256 of this review, the symbol \( \mathcal{H} \), which in the book signifies \( K \)-bi-invariance, was omitted everywhere it should have occurred, with considerable loss of meaning. The places where \( \mathcal{H} \) should have appeared are as follows:

- page 255 line 28 After \( \mathcal{E}'(G) \)
- page 255 line 31 After the second “and”
- page 255 line 33 After \( \mathcal{E}'(G) \)
- page 255 line 35 After \( \mathcal{D}(G) \)
- page 255 line 37 After \( \mathcal{E}'(G) \) (twice)
- page 255 line 43 After \( \mathcal{E}'(G) \)
- page 255 line 47 After \( \mathcal{E}'(G) \)
- page 256 line 2 After \( \mathcal{E}'(G) \)
- page 256 line 4 After \( \mathcal{E}'(G) \)

Also, in case any readers of the review were unsure as to whether the six properties of spherical functions, listed on page 255, are