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DEFORMATION RIGIDITY FOR SUBGROUPS OF $SL(n, \mathbf{Z})$ ACTING ON THE n-TORUS

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ABSTRACT. We announce and give a sketch of the proof of the result:

Theorem 1. *For $n \geq 3$, the standard action of $SL(n, \mathbf{Z})$ on $\mathbf{T}^n$ is smoothly and analytically rigid under C^0-deformations.*

Several related results concerning rigidity of actions of subgroups of $SL(n, \mathbf{Z})$ on $\mathbf{T}^n$ that follow from our method are also discussed.

§1. RIGIDITY THEOREM

The natural action of the determinant-one, integer $n \times n$-matrices $SL(n, \mathbf{Z})$ on $\mathbf{R}^n$ preserves the integer lattice $\mathbf{Z}^n$; hence for each subgroup $\Gamma \subset SL(n, \mathbf{Z})$ there is an induced standard action on the quotient n-torus, $\varphi: \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$. A basic problem is to understand the smooth actions near to such a standard action in terms of their geometry and dynamics (cf. [8, 19]). In this note we announce results which classify 1-parameter deformations of standard actions.

A C^k-deformation of φ is a 1-parameter family of C^∞-actions $\varphi_t: \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$, $0 \leq t \leq 1$ such that $\varphi_0 = \varphi$ and for each $\gamma \in \Gamma$, the C^∞-maps $\varphi_t(\gamma)$ depend C^k on the parameter t. That is, $\varphi_t(\gamma)$ is a C^k-path in the Frechet space $\mathrm{Diff}^\infty(\mathbf{T}^n)$. A C^k-deformation is *trivial* if it is implemented by a C^k-family of inner automorphisms of $\mathrm{Diff}^\infty(\mathbf{T}^n)$. That is, there exists a 1-parameter family of C^∞-diffeomorphisms $H_t: \mathbf{T}^n \to \mathbf{T}^n$ which depends C^k on the parameter, and for all $\gamma \in \Gamma$ satisfies

$$(1) \qquad H_t^{-1} \circ \varphi_t(\gamma) \circ H_t = \varphi(\gamma); \qquad 0 \leq t \leq 1.$$

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We say that φ_t is *locally trivial* if H_t as above exists for $0 \le t < \varepsilon$ for some $\varepsilon > 0$. The action φ is smoothly C^k-(*locally*) *rigid* if every C^k-deformation is (locally) trivial. When the deformation φ_t consists of real analytic maps, then we can require that each H_t be real analytic, which leads to the corresponding notion of *analytically* C^k-(*locally*) *rigid.*

Theorem 1. *Let $\Gamma \subset SL(n, \mathbf{Z})$ be a subgroup of finite index with $n \ge 3$. Then the standard action of Γ on $\mathbf{T}^n$ is smoothly and analytically C^1-rigid, and C^0-locally rigid.*

The conclusion of the theorem is false for $n = 2$: There exists finite-index subgroups of $SL(2, \mathbf{Z})$ freely-generated by two hyperbolic elements, and their action on $\mathbf{T}^2$ can be smoothly deformed using the examples of (§24, [1]) to a topologically equivalent action for which the generators are not smoothly conjugate to their standard action. (The obstruction is that the stable and unstable foliations for the deformed actions are not even C^2, hence are not conjugate to linear foliations [1, 9].)

The conclusion of the theorem can be reinterpreted as stating that every C^1-path, starting at the standard action, in the space of representations $\mathscr{R}(\Gamma, G)$ for $G = \mathrm{Diff}^r(\mathbf{T}^n)$ with $r = \infty$ or ω, is obtained by conjugating φ with a path in G starting from the identity.

The tangent space of $\mathbf{R}(\Gamma, G)$ at φ maps to the space of infinitesimal deformations of φ, which are identified with the first cohomology group $H^1(\Gamma; \mathrm{Vect}^\infty(\mathbf{T}^n))$. Here, $\mathrm{Vect}^\infty(\mathbf{T}^n)$ denotes the Frechet Γ-module of smooth vector fields on $\mathbf{T}^n$. J. Lewis proved in his thesis [10] that for $n \ge 7$ and Γ of finite-index, $H^1(\Gamma; \mathrm{Vect}^\infty(\mathbf{T}^n))$ is zero, so that the standard action is infinitesimally rigid. It is unknown whether this group must vanish for nonalgebraic actions of Γ, especially those which do not preserve a smooth volume form on $\mathbf{T}^n$.

The proof of Theorem 1 has two main steps, which are described in more detail in §§2,3 below. We first prove topological rigidity; that is, we show there exists a homeomorphism H_t of $\mathbf{T}^n$ satisfying (1) above. This result holds in much greater generality than Γ of finite index—see Theorem 2 below. We assume the existence of an element $\gamma \in \Gamma$ which acts on $\mathbf{T}^n$ as an Anosov diffeomorphism; then by the unique structural stability for toral Anosov diffeomorphisms [1, 5, 17] there is a unique candidate for H_t. Our main idea is to investigate the periodic points for $\varphi(\gamma)$ that are also periodic under the full group action $\varphi(\Gamma)$. We combine results of Margulis [15] and Stowe [18] to prove that the Γ-periodic points are stable under deformation, from which we easily obtain our conclusion (1).

The second step is to show that a topological conjugacy must be as regular as the actions—either smooth or analytic. This is much more delicate and technical, and essentially requires Γ to be of finite index, but applies to perturbations as well as to deformations. The methods are based on recent results in smooth hyperbolic dynamical systems [9, 12–14]. Let us mention that the foundation of the regularity theory for conjugacies is the Livsic theorem [11], which states that the cohomology class of an additive cocycle over an Anosov system is determined by its values at periodic orbits. Superrigidity of lattices in $SL(n, \mathbf{R})$ for $n \geq 3$ implies rigidity of the expansion coefficients of Γ at periodic orbits, which begins the proof of regularity of H_t. Thus, in both steps it is key to concentrate on the behavior of periodic orbits, which illustrates the general principle that the periodic orbits for hyperbolic dynamical systems control the global dynamics.

A C^1-perturbation of φ is an action $\varphi_1 : \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ so that for a given set of generators $\{\gamma_1, \dots, \gamma_d\}$ of Γ, the diffeomorphisms $\varphi(\gamma_i)$ and $\varphi_1(\gamma_i)$ are C^1-close in the uniform topology. It remains a basic open problem to show that every C^1-perturbation of a standard action is trivial. Equivalently, does the adjoint action of $G = \mathrm{Diff}^\infty(\mathbf{T}^n)$ on $\mathscr{R}(\Gamma, G)$ have open orbit at φ in the C^1-topology? By Theorem 3 below, it would suffice to show topological stability; that is, to prove that the orbit of $\mathrm{Homeo}(\mathbf{T}^n)$ at φ intersects $\mathscr{R}(\Gamma, G)$ in an open set.

§2. Topological stability

Let $\varphi : \Gamma \times X \to X$ be a C^∞-action of a finitely-generated group Γ on a compact manifold X of dimension n. The *periodic points* $\Lambda \subset X$ of φ are those $x \in X$ whose $\varphi(\Gamma)$-orbit is a finite set. An element $\gamma \in \Gamma$ is said to be hyperbolic if $\varphi(\gamma)$ is an Anosov diffeomorphism [1] of X. For example, if $\Gamma \subset SL(n, \mathbf{Z})$ and $X = \mathbf{T}^n$, then γ is hyperbolic if it has no eigenvalue of modulus 1. Moreover, if Γ contains a hyperbolic element, then the φ-periodic points Λ are exactly the rational points $(\mathbf{Q}/\mathbf{Z})^n$.

For $x \in \Lambda$, let Γ_x denote the stabilizer of the *orbit* of x and $\rho_x: \Gamma_x \to GL(T_x X)$ the isotropy linear representation. We say that an irreducible representation $\rho: \Gamma \to GL(\mathbf{R}^k)$ is *noncompact* if the algebraic hull H_ρ of $\rho(\Gamma)$ is a noncompact Lie group. More generally, if ρ is reducible then for each nontrivial invariant subspace $E \subset \mathbf{R}^k$ the restricted action ρ_E of Γ on E has noncompact algebraic hull. The action φ is *strongly infinitesimally rigid* at $x \in \Lambda$ if the representation ρ_x satisfies the vanishing cohomology condition:

(2) For all noncompact representations $\rho: \Gamma_x \to GL(k, \mathbf{R})$, the first cohomology group with coefficients in the ρ-module $\mathbf{R}^k$, $H^1(\Gamma_x; \mathbf{R}^k)$, vanishes.

Note that by a remarkable result of Margulis [15], condition (2) holds for lattices in $SL(n, \mathbf{R})$ for $n \geq 3$, as well as for many other lattices in higher rank semisimple Lie groups.

Theorem 2. *Let $\varphi: \Gamma \times X \to X$ be a smooth action of the finitely-generated group Γ satisfying*:

(a) *φ is strongly infinitesimally rigid at each $x \in \Lambda$,*
(b) *Λ is dense in X,*
(c) *there exists $\gamma_0 \in \Gamma$ hyperbolic.*

Then for each C^0-deformation φ_t of φ, there exists $\varepsilon > 0$ and a continuous family of homeomorphisms H_t of X defined for $0 \leq t < \varepsilon$ conjugating the action φ_t to φ.

Sketch of proof. Define a sequence of normal congruence subgroups $\Gamma_{p+1} \subset \Gamma_p \subset \Gamma$ for $p \geq 1$ of finite index with $\bigcap \Gamma_p = I$. For example, let Γ_p be the stabilizer subgroup for all $x \in \Lambda$ with orbit length $\leq p$. Or, for $\Gamma \subset SL(n, \mathbf{Z})$, we can let Γ_p be the matrices congruent to I mod $(p!)$. Let Λ_p be the fixed-point set of $\varphi(\Gamma_p)$. Clearly, each $\Lambda_p \subset \text{Periodic}(\varphi(\gamma_0))$. Let $\varepsilon > 0$ be such that $\varphi_t(\gamma_0)$ is Anosov for $0 \leq t < \varepsilon$ and H_t be the unique continuous family of homeomorphisms conjugating $\varphi_t(\gamma_0)$ to $\varphi(\gamma_0)$ for $0 \leq t < \varepsilon$ with $H_0 = \text{Id}$ (cf. [1, 17]). For each $x \in \Lambda$ set $x_t = H_t(x)$, which will be an isolated fixed-point for some power $\varphi_t(\gamma_0^m)$. The fundamental observation is then:

Lemma 2.1. *x_t is an isolated fixed-point for $\varphi_t(\Gamma_p)$ for $0 \leq t < \varepsilon$.*

Proof. The representation ρ_x of Γ_p at $x \in \Lambda_p$ satisfies Stowe's criteria [18] for unique local stability of the fixed-point of Γ_p near

x, so there is a unique fixed-point y_t for $\varphi_t(\Gamma_p)$ near x for t small. For $m \neq 0$ such that $\gamma_0^m \in \Gamma_p$, local uniqueness of x_t implies that $y_t = x_t$. Moreover, by condition (2) the hypotheses of Stowe's theorem are closed in the parameter t, so via a maximum argument x_t must be stable for all $t < \varepsilon$. □

The lemma implies that H_t conjugates $\varphi_t(\Gamma)$ to $\varphi(\Gamma)$ on the dense set Λ. By continuity of the actions, the conjugacy also holds for the closure X of Λ. □

If $\Gamma \subset SL(n, \mathbf{Z})$ and the deformation φ_t is C^1 in the parameter, then with more work one can show that $\varphi_t(\gamma_0)$ is Anosov for all $0 \le t \le 1$. Hence H_t exists for all $0 \le t \le 1$ also, and we can take $\varepsilon = 1$ above.

§3. Differentiable conjugacy

For $\Gamma \subseteq SL(n, \mathbf{Z})$ of finite index, the topological conjugacies exhibited in §2 are in fact smooth (or analytic) if φ_t is a smooth (or analytic) action, with C^1-dependence on the parameter for a C^1-deformation. The proof of this is based on the cohomological properties of Γ and recent advances in the theory smooth Anosov systems [9, 11–14].

Theorem 3. *Let $\Gamma \subseteq SL(n, \mathbf{Z})$ be of finite-index with $n \ge 3$. Suppose that φ_1 is a smooth (analytic) action that is topologically conjugate to the standard action of Γ on T^n, and C^1 close to it. Then the conjugacy is smooth (analytic). Moreover, if a 1-parameter C^0 (C^1)-deformation of the standard action is given, then the smooth conjugacy depends C^0 (C^1) on the parameter.*

Sketch of proof. Γ satisfies condition (2), so that for each $p \in \Lambda_p$ we have $H^1(\Gamma_p; \mathbf{sl(n, R)}) = 0$. (For $n \ge 5$, this vanishing follows also by the previous results of Borel [2, 3].) Consequently,

Proposition 3.1. *For each $x \in \Lambda_p$ the infinitesimal representation of Γ_p at x is stable under C^0-deformation.* □

The corollary of the proposition we need is that for any semisimple element $\gamma \in \Gamma$, with no eigenvalues of modulus one, the Anosov diffeomorphisms $\varphi_t(\gamma)$ have constant exponents in t at all of their periodic orbits, and this condition is closed in the parameter. From the results of [16], we can choose a maximal Abelian semisimple subgroup $\Gamma_{sa} \subset \Gamma$ that is free of rank $n-1$. A simultaneous diagonalization of this subgroup yields n transverse, one-dimensional foliations of $\mathbf{T}^n$ which are each invariant under the action of Γ_{sa}. (We say that the action of Γ on $\mathbf{T}^n$ is *trellised.*) By the topological stability of the action, these foliations persist under the deformation, and are denoted by $F_i(t)$ for $1 \leq i \leq n$.

Proposition 3.2. *For each i and t, the individual leaves of the foliation $F_i(t)$ are C^∞-immersed submanifolds of* $\mathbf{T}^n$.

Proof. For each i, there exists an element $\gamma_i \in \Gamma_{sa}$ such that $F_i(t)$ is the contracting foliation of the smooth Anosov map $\varphi_t(\gamma_i)$. The conclusion then follows from the theory of Hirsch, Pugh, and Shub [6]. □

It follows from Proposition 3.2 that each H_t maps $F_i(0)$ to $F_i(t)$, and conjugates two smooth Anosov diffeomorphisms for which these are the stable foliations. Therefore, by methods as in [12], H_t is smooth when restricted to the one-dimensional leaves of $F_i(t)$ for each i and t, so by the natural extension of the Regularity Theorem 1.3 of [9], the map H_t is itself smooth.

It remains to show that the maps H_t depend C^k on the parameter. For $k = 0$, this is a consequence of the continuous dependence of the C^∞-immersed stable manifolds of an Anosov diffeomorphism of the parameter. For $k = 1$, we remark that the Anosov diffeomorphisms chosen in the proof of Proposition 3.2 will have greater contraction exponent than any expansion exponent, as they preserve a smooth volume form by the stability of the exponents. Thus, the stable foliations are defined as C^1-structures via a contraction principle (cf. [9, 14]), which also implies that the conjugacy will depend at least C^1 on the data.

Details of the above proofs will appear in [7]. In addition, further applications to the rigidity of other group actions (cf. [4]) are discussed.

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