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*Derivatives, nuclei and dimensions on the frame of torsion theories*, by Jonathan S. Golan and Harold Simmons. Pitman Research Notes in Mathematics Series no. 188, Longman Scientific & Technical, Harlow, U.K. (co-published in the United States with John Wiley & Sons Inc., New York), 1988, 120 pp., $44.95. ISBN 0-582-03448-5

This monograph is remarkable not so much for the new results which it contains, as for the fusion of two hitherto separate traditions which it represents: the algebraic tradition of studying noncommutative rings via their module categories (and more particularly via localizations of the latter), and the lattice-theoretic study of frames and their nuclei, whose main inputs have come from logic and category theory (particularly topos theory).

In one sense, it comes as no surprise that these two traditions should have coalesced. As Borceux and Kelly [1] have shown, the localizations of any well-behaved category have a natural tendency to form a frame in their canonical ordering (incidentally, it seems odd that [1] does not appear among the references of the book under review). Also, in the representation theory of commutative rings the utility of frames is well understood (see [6, Chapter 5] for a survey; the key point is that in the commutative case one can pass directly from a ring to its frame of radical ideals and the other frames associated with it, without having to go by way of the module category). However, in the noncommutative case there has until recently been a noticeable lack of communication between ring theory and frame theory: the subjects have been advanced by disjoint sets of people (with the notable exception of J. Lambek; however, his work on rings [7] predates his interest in categorical logic, and does not use techniques from the latter), and have developed distinct traditions of terminology and notation. (Although the present book makes a start on bridging the latter gulf of misunderstanding, it seems a shame that the authors have not provided an index of notation.)

The two authors of the present book well represent the two traditions mentioned above. Golan has long been one of the foremost (and most productive) workers in noncommutative ring theory, and his two previous books [2, 3] will be well known to all who have studied the subject. Simmons, by training a logician, became interested in frame theory in the late seventies [8], and made significant contributions to the topological side of the subject, in particular an order-theoretic analysis of the Cantor–Bendixson derivative [9]. By the time [3] appeared, it had become clear that Simmons' techniques could also be usefully applied in the module-theoretic context being studied by Golan; the upshot was a visit by Simmons to the University of Haifa in May 1987, which provided the opportunity for the collaboration that led to this book.

The book reads like "work in progress," and is not self-contained: for example, the reader will not find the definition of a torsion theory here (although Gabriel filters are defined). Equally, there is little or nothing in the way of applications of the theory developed to particular rings or to particular ring-theoretic problems. Thus this is a work for the specialist, not for the general mathematical reader: if the latter wants to know about the significance of torsion theories as a tool for studying noncommutative rings he would be better advised (albeit with the reservations expressed by Hodges [4]) to plough through [3], and if he wants to know in more general terms what frames can do for him he should probably try [6].

Nonetheless, as a *Bulletin* reviewer, I am under instructions from the editors to try to provide the general reader with an account of the area to which this book belongs, and of the contribution which it makes to it. As regards *torsion theories*, I see neither the necessity nor the possibility of improving on Hodges' succinct account [4]; for the purpose of what follows, all the reader needs to know is that they form a particularly well-behaved class of localizations of the category of (left) modules over a given ring.

Taking the other (possibly) unfamiliar words in the book's title in reverse order: a *frame* is a complete lattice satisfying the infinite distributive law which is characteristic of the open-set lattices of topological spaces. Frames are extensionally the same as complete Heyting algebras (that is, complete lattices equipped with an implication operator $\rightarrow$ satisfying $a \leq (b \rightarrow c)$ iff $(a \wedge b) \leq c)$, but intensionally different: the point is that the natural homomorphisms to consider between frames preserve arbitrary joins and finite meets, but not the implication operator. As mentioned earlier, the localizations of a well-behaved category tend to form a frame in their natural ordering, and the torsion theories on the category of $R$-modules are no exception.

*Nuclei* on a frame correspond to localizations of the frame itself, considered as a category (that is, to its quotients in the category of frames); they are unary operators ( $j$, say) on the frame which are inflationary (i.e., $a \leq j(a)$ ) and idempotent $(j(a) = jj(a))$ and preserve finite meets. Yes (since you ask), the nuclei on a given frame $A$ themselves form a frame $N(A)$ (this is due to Isbell [5]); from the viewpoint that a frame is a generalized topological space, $N(A)$ can be thought of as a substitute for " $A$ retopologized with the discrete topology" (only a substitute, because $N(A)$ isn't in general a Boolean algebra—although it coincides with $A$ iff $A$ is Boolean).

In studying nuclei, one frequently encounters unary operators which satisfy two of the three conditions above but fail to be idempotent; these are called *prenuclei.* (A *derivative* is a still weaker notion, where the condition " $j$ preserves finite meets" is relaxed to " $j$ preserves order.") For example, the composite of two different nuclei is in general only a prenucleus. However, every prenucleus gives rise to a nucleus (and every derivative gives rise to a closure operator) if you iterate it enough times (transfinitely often, if necessary—taking joins at limit ordinals). It was Simmons' insight that the number of steps needed for this iteration to converge could, in particular cases, be regarded as a kind of dimension function of the frame and/or the prenucleus under consideration: the case treated in [9] was the Cantor–Bendixson rank of a scattered space, but subsequently [10] he observed that the Gabriel dimension of a module (and other module-theoretic dimension functions) could be treated in a way which is formally exactly the same.

The book under review is concerned with the further investigation of these ideas. Having set up the basic context of the frame of torsion theories, the authors' main concern is to introduce a bestiary (their word, not mine) of particular derivatives, prenuclei and nuclei on it which may be thought of as capturing some algebraic information about the underlying ring, and then to indicate how various familiar (and unfamiliar) notions of dimension for rings and modules may be extracted from this machinery, as described in the previous paragraph. As mentioned previously, no substantial applications of the machinery are given here: the book ends with the proof of a generalization of Stafford's stable range theorem [11], but no indication is given of what the utility of this theorem might be. Nevertheless, it would be surprising if the exploitation of all this machinery did not in due course lead to susbstantial advances in ring theory; so professionals in that field will want to familiarize themselves with the machinery by reading this book.

They will find it tolerably easy to do so, though the authors' style (principally the first author's, I imagine) is rather dry and relies heavily on the reader's ability to remember notation (as mentioned earlier, an index of notation would have been very helpful). One instance which this reviewer found irritating is the use, throughout Chapter 4, of the Roman letter w to denote a certain limit ordinal (defined on p. 92); this constantly looks as if it ought to be a Greek $\omega$, but isn't. The book has been produced with commendable speed on the part of both authors and publisher, but does seem to have suffered from slightly inadequate proofreading; there is a fair sprinkling of misprints, although most of them will not cause the average reader any trouble.

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*Algebraic homotopy*, by H. J. Baues. Cambridge University Press, Cambridge, New York, 1989, 466 pp., $89.50. ISBN 0-521-33376-8

The notion of classification of structure arises in many areas of mathematics, and a common classification is "up to homotopy," or in terms of "deformation." For this reason, techniques of homotopy theory, and in particular the fundamental group and higher homotopy groups, are important and have been applied across a range of mathematical disciplines.

*Algebraic Homotopy*, which we refer to as AH, has in the Introduction the following quotation from J. H. C. Whitehead's address to the International Congress of Mathematicians at Harvard in 1950 [W 7]:

> In homotopy theory, spaces are classified in terms of homotopy classes of maps, rather than individual maps of one space in another. Thus, using the word category in the sense of S. Eilenberg and Saunders Mac Lane, a homotopy category of spaces is one in which the objects are topological spaces and the 'mappings' are not individual maps but homotopy classes of ordinary maps. The equivalences are the classes with two-sided inverses, and two spaces are of the same homotopy type if and only if they are related by such an equivalence. The ultimate object of *algebraic homotopy* is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry.