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Andreotti-Grauert theory by integral formulas, by Gennadi M. Henkin and Jürgen Leiterer. Progress in Mathematics, vol. 74, Birkhäuser, Boston, Basel, Berlin, 1988, 270 pp., $44.90. ISBN 0-8176-3413-4

Experts in multidimensional complex analysis will find the title of this monograph sufficiently informative, but most other mathematicians will probably feel lost, and perhaps not bother to look closer at this book. That would be regrettable, because what is before us is the first attempt to make accessible to a wider audience the deep work of A. Andreotti and H. Grauert published in a classic paper in 1962 [AnGr]. The results and techniques developed in this paper have become standard fare for researchers in several complex variables (though they have not yet appeared in book form), and they still continue to exert a profound influence.

The theory of Andreotti and Grauert bridges the gap between the two extreme cases of complex manifolds for which complex analysis had been developed thoroughly by the mid-1950s, namely the compact ones on the one hand, and the so-called Stein manifolds on the other. (For the purposes of this review we will only consider complex *manifolds*, and ignore the fact that all results are in fact true for complex *spaces*, a generalization of complex manifolds which includes "analytic singularities.")

Stein manifolds were first introduced by Karl Stein in 1951 [Ste] as an abstract generalization of domains of holomorphy in $\mathbf{C}^n$; in contrast to compact manifolds they carry lots of nontrivial holomorphic functions, enough to make function theory on such manifolds a natural generalization of function theory on open subsets of the complex plane, or on noncompact Riemann surfaces. Stein manifolds can be characterized in many ways: for example, they are precisely those manifolds of positive dimension which can be embedded as closed submanifolds of some $\mathbf{C}^N$. Their most important analytic property is given by H. Cartan's Theorem B: On a Stein manifold M all cohomology groups $H^q(M, S)$ with values in a coherent analytic sheaf S vanish for $q \geq 1$. Moreover, this triviality of the analytic sheaf cohomology characterizes Stein manifolds. An excellent reference is the monograph by H. Grauert and R. Remmert [GrRe].

In contrast to real manifolds, *complex* compact manifolds do not exist in $\mathbf{C}^n$ (except for the trivial case of a finite set of points), so examples of such manifolds are necessarily more abstract. Among the simplest examples are the complex projective spaces $\mathbf{CP}^n$ and the complex tori. It is a deep theorem, due to Chow, that every complex submanifold of $\mathbf{CP}^n$ is algebraic, i.e., it can be defined as the common zero set of a finite set of homogeneous polynomials. In contrast to the situation in dimension one, in dimension ≥ 2 there exist complex manifolds that are not algebraic; such examples were already known to Riemann, who in fact determined the necessary and sufficient conditions for a complex torus (i.e. the quotient of $\mathbf{C}^n$ by a lattice) to be embeddable in $\mathbf{CP}^n$. A classical theorem of Cartan and Serre states that on a compact complex manifold X all coherent analytic sheaf cohomology groups $H^q(X, S)$ are finite dimensional (see [GrRe]).

In higher dimensions there exist of course lots of noncompact manifolds which are not Stein. The simplest examples are open sets in $\mathbf{C}^n$, $n \geq 2$, which are not domains of holomorphy, like the complement of a ball. Less trivial, but still natural examples arise by considering the complements in $\mathbf{CP}^n$ of complex submanifolds of codimension ≥ 2. In view of the results mentioned above, the obvious question then is: What can one say about the cohomology groups of such manifolds? In order to answer this meaningfully, one must consider some natural and verifiable conditions on such manifolds, which somehow lie between compact and Stein. Let us now consider the classes of manifolds which were introduced by Andreotti and Grauert. First, we recall that for a real valued C^2 function ϕ on an open set D in $\mathbf{C}^n$ the Levi form (or complex Hessian) is defined by

$$L(\phi; z, t) = \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k.$$

The Levi form is a Hermitian form in t, and ϕ is said to be strictly plurisubharmonic on D if the Levi form is positive definite at every point $z \in D$. More generally, ϕ is said to be q-convex, where $1 \leq q \leq n$, if at each point $z \in D$ the Levi form of ϕ has at least $n-q+1$ positive eigenvalues. ϕ is said to be q-concave if $-\phi$ is q-convex. Note that strictly plurisubharmonic functions are precisely the 1-convex functions. Moreover, q-convex functions remain q-convex under biholomorphic maps, so that q-convex and q-concave functions can be defined on complex manifolds. The notion of q-convexity was introduced first by W. Rothstein (Math. Ann. 129, 1955). It is a deep theorem, essentially due to Grauert [Gra 1 and DoGr], that a complex manifold M is Stein if and only if there is a C^2 strictly plurisubharmonic exhaustion function ϕ on M (i.e. the sublevel sets $M_c = \{x \in M : \phi(x) < c\}$ have compact closure in M for every $c \in R$). It is now quite natural to make the following

Definition. The complex manifold X is said to be q-convex, $1 \leq q$, if there are a C^2 exhaustion function ϕ on X and a compact subset K of X such that ϕ is q-convex on $X - K$. If K can be chosen empty, i.e. if ϕ is q-convex on X, X is said to be q-complete. X is said to be q-concave if there is an exhaustion function on X which is q-concave outside a compact set. Compact manifolds are also called 0-convex.

Since q-concave functions cannot take on (local) minima at interior points, there do not exist global q-concave exhaustion functions, thus there is no concave analogue of q-completeness. Notice that by Grauert's characterization mentioned above, X is Stein if and only if X is 1-complete.

It is easy to construct lots of explicit examples of q-convex domains in $\mathbf{C}^n$ for any q. The complement of a coordinate ball or, more generally, of a strictly pseudoconvex domain in a compact complex manifold is easily seen to be 1-concave. Much deeper is the fact, proved in 1970 by W. Barth [Bar], that the complement of a complex submanifold of codimension q in projective space is q-convex (this does no longer hold for analytic subvarieties, i.e. in the presence of singularities!). Y. T. Siu and the reviewer proved that a real submanifold M of $\mathbf{C}^n$ for which at each point p the maximal complex subspace of the (real) tangent space $T_p(M)$ has dimension $\leq q$ has a neighborhood basis of $(q+1)$-complete domains [RaSi].

We can now state the principal results of Andreotti and Grauert. Many of these results had already been announced by Grauert in 1959 [Gra 2], shortly after the publication of his ground-breaking solution of the Levi problem in 1958 [Gra 1].

Let X be a complex manifold, q an integer ≥ 0, and S a coherent analytic sheaf on X. Then

$$\dim_{\mathbf{C}} H^r(X, S) < \infty$$

a) for all $r \geq q$ if X is q-convex;
b) for all $r < \operatorname{dih}(S) - q$ if $q > 0$ and X is q-concave, where $\operatorname{dih}(S)$ denotes the homological dimension of S.

Furthermore, if $q > 0$ and X is q-complete, then

$$H^r(X, S) = 0 \text{ for } r \geq q.$$

We do not define the notion of homological dimension of a sheaf precisely. In essence, it measures how close the sheaf is to being locally free, i.e., isomorphic to the sheaf of germs of holomorphic sections of a holomorphic vector bundle; the homological dimension of a locally free sheaf equals the dimension of the underlying base manifold. Homological dimension has its origins in commutative algebra; the concept was first introduced and investigated systematically for sheaves in the paper of Andreotti and Grauert.

Stripped of all technical details, there are essentially two major steps in the proof of these results. The first part involves proving an isomorphism between the sheaf cohomology groups in the relevant dimensions for two sublevel sets X_b and X_c of the q-convex, respectively q-concave exhaustion function ϕ for $b < c$ sufficiently close to each other. This is done by the "bumping technique" introduced in [Gra 1]: one shows that there is an isomorphism if X_b is "bumped out" just a little bit near a given boundary point, and then one repeats this step finitely many times until one has covered the whole boundary. The finite dimensionality of the cohomology groups then follows by the argument first introduced by Cartan and Serre in the proof of the finiteness theorem on compact complex spaces, i.e., by an application of a version of Montel's theorem for analytic cohomology classes, and by the compactness theorem of L. Schwartz for Fréchet spaces. In the second step one must show that there is an isomorphism between the cohomology groups of X and of X_b for b sufficiently large; the key ingredient here is a very delicate version valid for cohomology classes of the classical Runge approximation theorem for holomorphic functions. It goes without saying that in the spirit of the 1950s/early 1960s, the proofs involve the full machinery of coherent analytic sheaf cohomology. As of today, no substantial simplification of the original proofs is known which gives the results of Andreotti and Grauert in full strength, i.e., not just on manifolds, but on complex spaces.

Beginning in 1969, methods of integral representations were introduced on strictly pseudoconvex domains, and rapidly found wide applications to the solution of numerous problems in multidimensional complex analysis which could not be tackled by the classical sheaf theoretic methods. This is not the place to review these developments in detail. But suffice it to say that these methods were pioneered by H. Grauert and I. Lieb, who used a new construction of Cauchy-type kernels due to E. Ramirez, and, independently, by G. M. Henkin, one of the authors of the book under review. Moreover, it is by now well known that these methods can be used very effectively to build up the fundamental global analytic theory of several complex variables, thus providing alternatives to the sheaf theoretic methods and to the $L^2\,\overline{\partial}$-methods of L. Hörmander. Such expositions can be found in the earlier book by the present authors [HeLe], and in a book by the reviewer [Ran], to which the reader is referred to for more complete references.

In the book under review, Henkin and Leiterer write another major chapter in the program to build up several complex variables by methods of integral representations. They present an essentially self-contained new proof of the results of Andreotti and Grauert and of many of the subsequent variations, generalizations, and applications due to other mathematicians, by systematically using integral representations. These methods are quite a bit more elementary and easier than the original proofs, but they only give the classical results for the case of complex manifolds (i.e. no singularities) and for holomorphic vector bundles (i.e. locally free sheaves). These restrictions are in line with current trends in complex analysis, which are much more concerned with detailed information about the boundary behavior of analytic objects than with developing the foundations in the appropriate most general setting. In fact, the proofs in the book before us yield a lot of additional information involving precise estimates up to the boundary which are not available by the classical methods.

The philosophy of the proofs is similar to the classical one. First one studies the sublevel sets X_c of the given exhaustion function ϕ. The heart of the matter is a precise optimal estimate in Lipschitz norm for solutions of the Cauchy-Riemann equations on forms of type $(0, r)$ under suitable convexity/concavity hypotheses. This result involves a well-known extension of the Grauert–Lieb/Henkin construction of integral solution operators for $\overline{\partial}$ in the strictly pseudoconvex case. For the q-convex case, this was done first in 1974 by W. Fischer and I. Lieb [FiLi], and, independently, by Siu and the reviewer [op. cit.], while the key observation to handle the concave case is due to M. Hortmann [Hor], who did the 1-concave case in 1976; a few years later Lieb extended Hortmann's method to the q-concave case. The present authors introduce some technical improvements, for example, they allow certain nonsmooth boundaries, and add to the local estimates for $\overline{\partial}$ a local approximation theorem for $\overline{\partial}$-closed forms, which is needed later to pass by an exhaustion procedure from the sets $X_c = \{x: \phi(x) < c\}$ to the full manifold X. To derive from the precise local theorem the global finiteness results for X_c, the bumping technique is replaced by the smoothing property of the local integral solution operator for $\overline{\partial}$, which yields a compact operator from local $\overline{\partial}$-cohomology classes into itself. This can be globalized in a straightforward manner, and the finiteness statement is obtained from standard results about compact operators in Banach spaces. The vector bundle case is reduced to the case of $(0, q)$ forms via local trivializations and Dolbeault's isomorphism.

Even though the book by Henkin and Leiterer is technical and written in a rather dry no-nonsense style, the authors have been careful and precise, making the product quite readable and clear. The material was obviously written specifically for this book, in contrast to the authors' earlier book [HeLe], which for the better half reproduced verbatim material published in research journals. On the other hand the references and historical notes are not as accurate as in their earlier work—for example, some of the key references quoted above are missing. It is quite annoying—at least to this reviewer—that the authors decided to change the standard terminology. As defined by Henkin–Leiterer, a function is q-convex if its Levi form has at least q positive eigenvalues, thus making strictly plurisubharmonic functions n-convex (if defined on an n-dimensional complex manifold). Additional changes appear in the definition of q-convex/concave manifolds. Not only is this bound to cause confusion, but it makes the conclusion in all main theorems depend explicitly on the dimension n of the manifold. The index is too short, but the list of symbols is adequate and quite useful when reading this technical material. There is also a short list of open problems. At \$44.90 (for the nonsocialist countries edition) the price is exorbitant for a book under 300 pages which is duplicated from a typed manuscript (including many handwritten symbols). If you have a friend in a socialist country, enlist his help in getting you a copy of the edition published by Akademie-Verlag in Berlin (DDR); except for a soft cover, it is identical to the Birkhäuser edition, but it is much cheaper.

Henkin and Leiterer have to be thanked for having taken on the hard work to present a different and more elementary approach to the key ideas of the work of Andreotti and Grauert, even if this does not yield the full strength of the original results. Moreover, the best is still to come. As the authors state in the Introduction, this book is preparing the grounds for an additional book which will contain substantial applications of the Andreotti–Grauert theory *with estimates* to the theory of holomorphic vector bundles, the theory of the tangential Cauchy–Riemann equations, the Radon–Penrose transformation, and to inverse scattering problems. Rudiments of these applications are contained in several recent articles by Henkin and various of his collaborators (precise references are given in the book). I think we have much to look forward to in the promised future book by Henkin and Leiterer, especially if it will not just reproduce published research articles. Substantial new applications will turn the book under review from just being an interesting display of the power and versatility of the methods of integral representations in multidimensional complex analysis into an important fundamental reference which has the potential to exert a profound and long lasting influence on future research comparable to the original work of Andreotti and Grauert.

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