15. H. Le Dret, *Structure of the set of equilibrated loads in nonlinear elasticity and applications to existence and nonexistence*, J. Elasticity **17** (1987), 123–141.

16. J. E. Marsden and T. J. R. Hughes, *Topics in the mathematical foundations of elasticity*, in Nonlinear analysis and mechanics: Heriot-Watt symposium, Vol. 2, Pitman, London, 1978, pp. 30–285.

17. ——, *Mathematical foundations of elasticity*, Prentice-Hall, Englewood Cliffs, 1983.

18, F. Stoppelli, *Un teorema di esistenza e di unicità relativo alle equazioni dell' elastostatica isoterma per deformazioni finite*, Ricerche Mat. **3** (1954), 247–267.

19. C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, in Handbuch der Physik, Vol. III/3 (S. Flügge, Editor), Springer-Verlag, Berlin, 1965, pp. 1–602.

20. T. Valent, *Local theorems of existence and uniqueness in finite elastostatics*, in Finite elasticity (D. E. Carlson and R. T. Shield, Editors), Nijhoff, The Hague, 1978, pp. 401–421.

21. ——, *Pressure boundary problems in finite elasticity. Results on local existence, uniqueness and analyticity*, in Proceedings, meeting on finite thermoelasticity (G. Grioli, Editor), Accademia Nazionale dei Lincei, Roma, 1986, pp. 241–264.

22. T. Valent and G. Zampieri, *Sulla differenziabilità di un operatore legato a una classe di sistemi differenziali quasi-lineari*, Rend. Sem. Mat. Univ. Padova **57** (1977), 311–322.

23. C.-C. Wang and C. Truesdell, *Introduction to rational elasticity*, Noordhoff, Groningen, 1973.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Heat kernels and spectral theory, by E. B. Davies. Cambridge University Press, Cambridge, 1989, 197 pp., $49.50. ISBN 0-521-36136-2

For the usual Laplacian Δ in Euclidean space $\mathbf{R}^n$ and its spectral theory, there is a tremendous amount of information available, largely because of a number of explicit formulas that are known. For example, the heat semigroup $e^{t\Delta}$ (the operator that solves the heat equation $\partial u/\partial t = \Delta u$ for $t > 0$ from the initial value $u(x, 0)$, given certain weak growth conditions) is a convolution operator with kernel $(4\pi t)^{-n/2}e^{-|x|^2/4t}$. From the explicit formula for the kernel we can read off certain qualitative properties of the heat semigroup, for example, the estimate $\|e^{t\Delta} f\|_\infty \leq (4\pi t)^{-n/2}\|f\|_1$, or the fact that $u(x, t) = e^{t\Delta} f(x)$ is C^∞ in $t > 0$. Are these qualitative properties merely lucky accidents or do they arise from deeper and more robust sources? As long as we stick to the usual Laplacian on all of $\mathbf{R}^n$, we have no hope of answering this question; indeed it is not even clear that the question makes any sense. Fortunately, mathematicians are rarely content with sticking to the tried and true, and are forever seeking generalizations, and this has been especially true when it comes to Laplacians. Nowadays, Laplacians come in more flavors than Baskin-Robbins ice cream. For example, consider an operator of the form

$$L = \sum_j \frac{\partial}{\partial x_j}\left(\sum_k a_{jk}(x)\frac{\partial}{\partial x_k}\right)$$

on a domain $\Omega \subseteq \mathbf{R}^n$ for which there exist positive constants α, β such that $\alpha I \leq a_{jk}(x) \leq \beta I$ as $n \times n$ matrices for all $x \in \Omega$. In what follows we use the term *generalized Laplacian* to refer to this example in particular, and in a loose sense to other generalizations as well, such as the Laplace–Beltrami operator on a complete Riemannian manifold.

The general problem we have raised is the following: To what extent do results about the spectral theory of the usual Laplacian extend to the generalized Laplacians? These questions have been the subject of intense investigation over a period of many decades by a large number of mathematicians using a wide variety of techniques. The accumulated knowledge is quite impressive, but there are still many open problems and this area of research remains active.

What do I mean by "spectral theory"? To begin with, we need to obtain a self-adjoint realization of the generalized Laplacian. The best way to do this is to consider the associated quadratic form

$$Q(f, g) = \sum_{j,k} \int_\Omega a_{jk}(x) f(x) \overline{g(x)}\, dx,$$

take the closure of this quadratic form on $C^\infty_{\text{com}}(\Omega)$, and pass to the associated self-adjoint operator. This is the Friedrichs extension of L, or the Dirichlet operator. Informally, this realization restricts the domain of L to functions vanishing on the boundary, and this is exactly what it is if the boundary is regular. But the

Friedrichs extension exists regardless of the nature of Ω. There exist other realizations of L, usually connected with other boundary conditions, but for simplicity we discuss only the Dirichlet realization. (The Laplace–Beltrami operator on a complete Riemannian manifold is essentially self-adjoint, which means there is a unique self-adjoint realization.)

Once we have selected the self-adjoint realization of L, the von Neumann spectral theorem applies. The first set of questions concerns the nature of the spectrum — is it discrete, continuous, a mixture of both? If it is discrete, what can be said about the eigenvalues and the eigenfunctions? Typically, if the domain Ω is bounded, we expect a discrete spectrum, with an asymptotic expression for the eigenvalues, and estimates on the lowest nonzero eigenvalue λ_0 (usually we describe the eigenvalues by the equation $Lf + \lambda f = 0$ so they are positive) related to the geometry of Ω. The eigenvalue λ_0 has multiplicity one, and the associated eigenfunction, called the *ground state*, is positive on Ω.

Along with the spectral theorem comes a functional calculus that allows us to define functions of L. We have already mentioned the heat semigroup e^{tL}. There is also the Poisson semigroup $e^{-t(L)^{1/2}}$, the Schrödinger group e^{itL}, the wave equation propagator $\cos t\sqrt{-L}$, the resolvant $(\lambda I + L)^{-1}$. Each of these particular functions of L has its own interpretation and interest. There are many interrelationships between them, so that information about one of them can be used to obtain information about the others. For example, from the wave equation propagator we can synthesize other functions via

$$f(-L) = \int_0^\infty \cos t\sqrt{-L}\hat{f}(t)\,dt$$

if f is an even function. The Schrödinger operator can be obtained from the heat semigroup by analytic continuation. But there is no guarantee that you will get exactly the information you want about one of these functions of L even if you know all the most intimate secrets about another one.

In addition to questions about specific functions of L such as those listed above, we may also ask about classes of functions, for example, if f satisfies the pseudodifferential operator symbol conditions

$$|f^{(k)}(x)| \le c_k(1 + |x|)^{m-k} \qquad \text{for all } k,$$

then we expect that $f(L)$ should behave like a pseudodifferential operator of order $2m$.

In an area with such a diversity of problems, it is not surprising that a great diversity of techniques have been explored, and no one technique has emerged as dominant (one that usually leads first to the best possible result). Ideas from probability theory, pseudodifferential operators, Fourier integral operators, abstract functional analysis, and differential geometry all have an important role to play in penetrating and perfecting this area.

Among the many points of view, the one presented in the book under review may be called the "heat kernel approach." This point of view stresses the importance of obtaining the most detailed information about the heat kernel, and then using this information to solve other problems. The heat kernel approach has not always been successful in obtaining best possible results, even for studying the heat kernel itself. But a major breakthrough occurred in the mid 1980s in the introduction of Log–Sobolev techniques by Davies and Simon [DS], and the subsequent development of these techniques, largely by Davies and coworkers. As a consequence, the heat kernel approach has become a powerful and accurate technique that can hold its own in comparison with other approaches.

Log–Sobolev inequalities first arose in quantum field theory in the work of Federbush [F] and Gross [G], inspired by Nelson's work on hypercontractivity [Ne]. There is now a large amount of literature on the subject [DGS]. In the context of generalized Laplacians, a Log–Sobolev inequality is an estimate

$$(*) \qquad \int_\Omega f^2 \log f\, dx \leq \varepsilon Q(f, f) + \beta(\varepsilon)$$

for a suitable class of functions f with $\|f\|_2 = 1$ for $0 < \varepsilon < \infty$, where $\beta(\varepsilon)$ is a specified function. The key idea of Davies and Simon is that a Log–Sobolev inequality is almost equivalent to an estimate

$$(**) \qquad \|e^{tL} f\|_\infty \leq e^{M(t)} \|f\|_2$$

for the heat semigroup, for a specified function $M(t)$. More precisely, starting with $(*)$ and $\beta(\varepsilon)$ one obtains $(**)$ with $M(t)$ derived from $\beta(\varepsilon)$, or starting with $(**)$ and $M(t)$ one obtains $(*)$ with $\beta(\varepsilon)$ derived from $M(t)$. There is a small loss involved in going from $(*)$ to $(**)$ and back to $(*)$, or vice versa, but for many important examples the result is sharp except for a constant.

What is the significance of this almost equivalence? The estimate $(**)$ gives immediate information about the heat kernel, but it depends on the generalized Laplacian in a way that is not transparent. The estimate $(*)$, on the other hand, is rather far removed from the heat kernel, but it is clearly monotonic in the coefficients of the generalized Laplacian. Thus if you know $(**)$ for one generalized Laplacian, you can pass to $(*)$, use monotonicity to switch to another generalized Laplacian, and then get back to $(**)$ for the new Laplacian. Notice that smoothness of coefficients does not really play any role in Log–Sobolev inequalities, so this technique allows us to obtain information about heat kernels of generalized Laplacians with coefficients that are only measurable functions, in a seemingly effortless way!

The Log–Sobolev technique lies at the heart of many of the results given in the book under review. The presentation is clear and concise (an amazing amount of information is packed into under 200 pages, and it is largely self-contained). The results are elegant and often best possible. In addition to presenting his own work, Davies includes very readable accounts of two important recent results: the parabolic Harnack inequality of Li and Yau [LY], and the Gaussian lower bounds for heat kernels of Fabes and Stroock [FS] based on the ideas of Nash [Na]. All in all, this book provides a lively and timely account of important work, and will be a valuable resource for anyone interested in research in this area.

This book is very much a report from the research frontiers. Since this is a rapidly developing area, no work can represent the last word, and so let me conclude by discussing some work which has been done since this book appeared, but which is very relevant to its message. For the Laplace–Beltrami operator on a smooth compact n-dimensional Riemannian manifold, the estimate

$$\|u_\lambda\|_\infty \le c(1+\lambda)^{(n-1)/4}\|u_\lambda\|_2,$$

where $\Delta u_\lambda = -\lambda u_\lambda$, was recently established by Sogge ([S 1, S 2, S 3]) using Fourier integral operator techniques. These techniques depend heavily on the smoothness of the metric. Using heat kernel methods, Davies [D] obtains

$$\|u_\lambda\|_\infty \le c(1+\lambda)^{n/4}\|u_\lambda\|_2.$$

What is the reason for the discrepancy in the power of $1+\lambda$? The heat kernel methods yield a weaker result because they apply to a broader class of examples, where smoothness of the metric is not required. In fact Davies [D] shows by example that Sogge's estimate does not hold with a constant that is independent of the modulus of continuity of the metric, and the heat-kernel–derived estimates are close to being best possible in this greater generality.

Perhaps we can discern a moral here, analogous to the Heisenberg uncertainty principle in quantum mechanics — it is impossible to achieve the ultimate in sharpness and generality simultaneously.

REFERENCES

[D] E. B. Davies, *Spectral properties of compact manifolds and changes of metric*, Amer. J. Math. (to appear).

[DGS] E. B. Davies, L. Gross and B. Simon, *Hypercontractivity:a bibliographic review*, Proceedings of the Hoegh Krohn Memorial Conference.

[DS] E. B. Davies and B. Simon, *Ultra-contractivity and the heat kernel for Schrödin ger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), 335–395.

[FS] E. B. Fabes and D. W. Stroock, *A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash*, Arch. Rational Mech. Anal. **96** (1986), 327–338.

[F] P. Federbush, *A partial alternate derivation of a result of Nelson*, J. Math. Phys. **10** (1969), 50–52.

[G] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), 1061–1083.

[LY] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **56** (1986), 153–201.

[Na] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954.

[Ne] E. Nelson, *A quartic interaction in two dimensions*, Mathematical theory of elementary particles, MIT Press, Cambridge, Mass., 1966, pp. 69–73.

[S 1] C. D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), 43–65.

[S 2] ——, *On the convergence of Riesz means on compact manifolds*, Ann. of Math. (2) **126** (1987), 439–447.

[S 3] ——, *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–138.

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