BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Stochastic processes with a multidimensional parameter, by M. Dozzi. Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, England, 1989, 195 pp., $49.95. ISBN 0-582-03127-3

The general theory of stochastic processes with a multidimensional parameter began to emerge and develop in the 1970s, with the two pioneering works of R. Cairoli and J. B. Walsh [1], and of E. Wong and M. Zakai [4]. In order to understand this subject, let us recall that the general classical theory (see, for example, Dellacherie and Meyer's book [2]) deals with stochastic processes which are indexed by time, namely by a subset of the real line. Moreover, many of the basic tools of the theory depend on the total order structure of the parameter set. This is the case, for example, for the definition of a martingale, a Markov process, and the concept of stopping time. The fundamental difference between the classical theory and this one is the lack of a total order structure in the parameter set. In the multidimensional theory, the total order is replaced by the partial order induced by the Cartesian coordinates of the $\mathbf{R}^n$ space $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ when $x_i \leq y_i$, for all $i = 1, \dots, n$.

The extensions of the different tools in this context yield very interesting results. For example, in the classical theory an $\mathscr{F}$-martingale $\{X_t\}$ can be defined equivalently by two points of view: either by writing $E[X_t/\mathscr{F}_s] = X_s$ for all $s \leq t$, or by $E[X(s, t]/\mathscr{F}_s] = 0$, where $X(s, t] = X_t - X_s$ is the increment of the martingale on the interval $(s, t]$. These two equivalent formulas are generalized in the multidimensional case to two different concepts. The first one is called a martingale and the second one is called a weak-martingale. Moreover, following the kind of past you wish to consider, other types of martingales are obtained.

It turns out that all the kinds of martingales are necessary for the development of the theory: maximal inequalities, decompositions, optional sampling theorem, stochastic integrals, Ito's formula, stochastic differential equations, etc.

An interesting property of the filtrations is the following: Let $\{\mathscr{F}_t, t \in \mathbf{R}^n\}$ be an increasing family of σ-algebras of events and denote by $\mathscr{F}_t(C)$ the σ-algebra generated by the σ-algebras $\mathscr{F}_{t'}$, where $t_i' \leq t_i$ for all $i \in C$, where C is any subset of $\{1, \dots, n\}$. We say that the filtration satisfies the conditional independence property if for all bounded random variables X, all $t \in \mathbf{R}^n$ and $C \subset C' \subset \{1, \dots, n\}$,

$$E[X/\mathscr{F}_t(C')] = E[E[X/\mathscr{F}_t(C)]/\mathscr{F}_t(C' \setminus C)].$$

This property implies that the conditional expectations with respect to $\mathscr{F}_t(C)$ and $\mathscr{F}_t(C' \setminus C)$ commute, and is in fact very natural. Indeed the filtration generated by the Brownian sheet, as well as the filtration generated by the Poisson process, satisfies this property. Moreover, in the case $n = 2$, this property is equivalent to the following: For every integrable random variable X and for every two points s and t on the plane, we have $E[E[X/\mathscr{F}_s]/\mathscr{F}_t] = E[X/\mathscr{F}_{\min(s,t)}]$. In many cases, this property permits us to solve multidimensional problems by solving them in one dimension and iterating. However, this property disappears when we change the probability (even by an equivalent one); and therefore we always try to prove results without using it.

Returning to martingales, we discover many unexpected phenomena such as martingales which are bounded in L_p for all $p > 0$ and which do not converge almost surely. Let us mention also the fact that in the Doob–Meyer decomposition for a submartingale, we can obtain only a weak martingale.

Another interesting feature is the representation theorem, called also the Wong–Zakai formula [4], which says that a martingale with respect to a Brownian sheet filtration is a sum of two stochastic integrals. The first one is a simple extension of the ordinary stochastic integral, but the second one is a double stochastic integral defined in $\mathbf{R}^n \times \mathbf{R}^n$ for strong martingales.

An additional problem in the multidimensional theory is to extend well the "stopping time" concept. Here too, this notion has two important extensions: the stopping point and the stopping line which divides the parameter set into two regions.

For the sake of simplicity, most of the authors in this theory work and write their research papers in the two-parameter setting and leave to the reader the extension for more than two parameters. Dozzi succeeded in presenting the theory in the general space $\mathbf{R}^n$, $n \geq 2$, and found a not-so-cumbersome notation, I would say — almost an elegant notation — which permits a clear reading.

This book treats several aspects of the theory: The first chapter is devoted to stochastic integration and the second chapter is about stochastic differential equations. Together they present a very complete view of the current theory. The last chapter treats different problems such as Harnesses, some Markov properties, and local time.

The only criticism I find is that the title of this book is badly chosen and may deceive the potential reader into thinking that at least all the important subjects of the theory will be mentioned or treated. In fact only a few topics are included, and therefore a title such as "Selected chapters in..." would be preferable. Some parts of the theory which are lacking partially or entirely are Markov processes, filtering, optimal stopping, point processes [3], and a study of the Brownian sheet. In addition, some notions are used without definitions (for example, the predictable projection and the dual predictable projection of a process). I found very few misprints.

In spite of these minor reservations, I read this book with great pleasure and I warmly recommend it for everyone who is interested in this lovely theory.

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AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Art gallery theorems and algorithms, by Joseph O'Rourke. Oxford University Press, 280 pp., $45.00. ISBN 0-19-503965-3

Victor Klee once asked how many guards are sufficient to guard the interior of an art gallery room with n walls and which contains