about stochastic differential equations. Together they present a very complete view of the current theory. The last chapter treats different problems such as Harnesses, some Markov properties, and local time.

The only criticism I find is that the title of this book is badly chosen and may deceive the potential reader into thinking that at least all the important subjects of the theory will be mentioned or treated. In fact only a few topics are included, and therefore a title such as "Selected chapters in..." would be preferable. Some parts of the theory which are lacking partially or entirely are Markov processes, filtering, optimal stopping, point processes [3], and a study of the Brownian sheet. In addition, some notions are used without definitions (for example, the predictable projection and the dual predictable projection of a process). I found very few misprints.

In spite of these minor reservations, I read this book with great pleasure and I warmly recommend it for everyone who is interested in this lovely theory.

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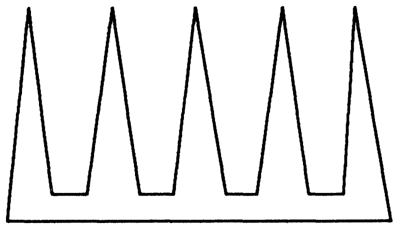
Art gallery theorems and algorithms, by Joseph O'Rourke. Oxford University Press, 280 pp., $45.00. ISBN 0-19-503965-3

Victor Klee once asked how many guards are sufficient to guard the interior of an art gallery room with n walls and which contains



no interior obstructions. It is easy to see that some art galleries require very few guards whereas some require a lot. The example of the comb-shaped gallery in the figure can readily be extended to give examples of galleries with n walls that require $\lfloor n/3 \rfloor$ guards, for any integer $n \geq 3$. In fact, no galleries with n walls ever require more guards. Vaĕsek Chvátal established this result, now known as the Art Gallery Theorem, by an argument that was largely combinatorial. Some years later, Stephen Fisk gave a different proof, which was also mostly combinatorial, and that we outline here.

Firstly, a floor plan of an art gallery room is an example of a *polygon* P: a collection of n vertices $v_1, \dots, v_n$ and n edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$ such that no pair of nonconsecutive edges intersect. Fisk's idea is first to divide the polygon up into triangles by adding nonintersecting chords to the polygon. Then $\lfloor n/3 \rfloor$ vertices are chosen in such a way that each triangle contains at least one of the chosen vertices. The guards are placed at the chosen vertices. Since each triangle has a guard, the entire polygon is guarded.

It remains to show how we can find the distinguished set of vertices. Suppose we could assign one of three colors to each of the vertices in such a way that no two vertices of the same color are joined by an edge. Every triangle would therefore have to receive each of the three colors. The least frequently used color gives us the required vertex set. The fact that any triangulated polygon can be three colored follows from a simple inductive argument: split the polygon into two triangulated polygons along any chord, color the two parts inductively, and paste together the polygon, relabeling the colors in one part if necessary.

The Art Gallery Theorem is a good example of the interplay between discrete and computational geometry, and it is this interplay that is the theme of O'Rourke's delightful book. Discrete geometry has been with us for some time, but computational geometry is a rather new branch of computer science. It is concerned with designing efficient algorithms for geometric problems, especially those in low dimensions involving points, lines, circles, planes, etc. Typical computational problems arising in the art gallery problem are: how quickly can a polygon be triangulated and how efficiently can a three coloring of a triangulated polygon be found. Other related computational problems include: given a polygon P, find the minimum number of guards that together guard all of P, and given a point in P, compute the region of P visible from the point.

By changing the original problem a little, we discover a wealth of interesting problems in both discrete and computational geometry. For example, we might note that buildings often tend to have corners which are either a right angle or three right angles. The corresponding polygons are called orthogonal. How many guards are required to guard orthogonal polygons? It turns out that $\lfloor n/4 \rfloor$ guards are always sufficient and sometimes necessary. This theorem is proved in the second chapter of the book, and algorithms for finding such a guard set are given. Both the proof and the algorithm are surprisingly more complex than for the unrestricted problem.

Another generalization is the possibility of mobile guards. The corresponding geometric notion is that of visibility from an edge. Several possible notions of edge visibility suggest themselves. Perhaps the most interesting is so-called weak visibility, which is the union of the regions visible from each point on the edge. Other generalizations include polygons with holes and exterior visibility. Going in the other direction, we might wonder if things get easier for special classes of polygons such as star-shaped, spiral, or monotone polygons, Chapter 3 to 6 deal with problems of this sort.

The next two chapters deal with visibility in polygons. A natural graph to study is the so-called visibility graph of polygon: the vertices of the graph are the vertices of the polygon and two vertices are adjacent in the graph if the line segment joining them in the polygon does not intersect the exterior of the polygon. It turns out that recognizing when a graph is a visibility graph of a polygon is a difficult and unsolved problem. These graphs are the topic of Chapter 7. Chapter 8 covers visibility from a computational geometry perspective: how quickly can we compute the visible region from a point or an edge for example? It turns out that linear time algorithms are available for the first problem but not for the second, although the latter can be solved in $O(n \log \log n)$ time, which is pretty fast as these things go.

The computational results are not all so encouraging. In Chapter 9, computationally hard problems are discussed. For example, it is NP-hard to find the minimum number of guards required to guard a polygon with n vertices, with or without holes. Some restricted cases of the problem are solvable, and these are discussed. The minimum guarding problem can be considered as a minimum covering of a polygon with star-shaped pieces. Although this is NP-hard, the problem of finding a minimum partition into star-shaped pieces is solvable in polynomial time, provided that each piece is star-shaped from a vertex of the polygon. If we allow holes however, the problem becomes NP-hard.

In the last chapter, we get a glimpse at three dimensions. The reader gets several surprises here. One of which is the fact that (nonconvex) polyhedra exist with n vertices that require $cn^{3/2}$ guards, so that the obvious procedure of placing a guard at each vertex does not guarantee that the entire polyhedron is guarded.

Since the book was published in 1987, there have been many new developments. The open problem mentioned on page 217 concerning the existence of an algorithm for finding a visibility graph in time proportional to the number of its edges, m, has been essentially solved. Hershberger [3] found an algorithm for finding the visibility graph in $O(e + n \log \log n)$ time. Two Ph.D. theses have appeared on the subject which contain substantial new results obtained since the book appeared. Everett [2] has found some new classes of polygons whose visibility graphs can be recognized in polynomial time and shows that the recognition problem is in PSPACE. She relates the visibility graphs of certain restricted classes of polygons to well-known classes of graphs. On the other hand, she shows that there is no finite set of forbidden induced subgraphs which characterize visibility graphs and disproves a conjecture of Ghosh on the sufficiency of a set of necessary conditions for visibility graph recognition. Shermer [4] has extended many of the results given in the book to the case of link-j visibility and to guard classes that are L_k convex. Two vertices of a polygon are link-j visible if they can be joined by a path of at most j edges that lies completely inside the polygon. A region is L_k convex if every two points in the region are link-k visible. A typical result obtained by Shermer is that the art gallery theorem generalizes to link-j visibility and L_k convex guard classes by replacing $\lfloor n/3 \rfloor$ by $\lfloor n/(k+2j+1) \rfloor$ (note that for normal visibility and point guards, $k = 0$ and $j = 1$). Czyzowicz, Rivera-Campo, Santoro, Urrutia, and Zaks have looked at a variant of perhaps more interest to architects: galleries with rooms. They show that a rectangular gallery divided into n rectangular rooms, such that there is a door between every pair of adjacent rooms, can be guarded with $\lfloor n/2 \rfloor$ guards [1]. There are many other related results too numerous to mention here.

In summary, this book succeeds on a number of levels. It is suitable for use as an introduction to computational and discrete geometry accessible to undergraduates. It is also a pleasant introduction to computational geometry for the outsider. By taking a single topic and treating it comprehensively, many of the important issues in the field are discussed without the maze of definitions that would be necessary to treat the whole of computational geometry. The reader is left with many unsolved questions to think about, along with the author's remark that he "would be disappointed if many of the unsolved problems in this book are not solved in the next decade." It is, after all, a rather young field.

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