To handle the inevitable ambiguity of sign in $SL(2, \mathbf{C})$, the author orients his lines so that if l has ends u and u' taken in that order it corresponds to the half-turn matrix

$$\mathbf{l} = \frac{i}{u' - u}(u + u' \quad -2uu'\partial \quad -u - u').$$

Then trace relations for products of these matrices based on the formula

$$\operatorname{tr} \mathbf{a} \operatorname{tr} \mathbf{b} = \operatorname{tr} \mathbf{ab} + \operatorname{tr} \mathbf{a}^{-1}\mathbf{b}$$

give the desired trigonometrical relations without ambiguity of sign. Full details, including conventions to handle special position and degeneracy and additional machinery to handle opposite isometries, must await the reader's own study of this intriguing book. For a first perusal that quickly reaches the most accessible parts of the main results, I recommend §§I.3, V.3, VI.2, and VI.5 and 6.

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The classical groups and K-theory, by A. J. Hahn and O. T. O'Meara. Springer-Verlag, Berlin, New York, 1989, 565 pp., $119.00. ISBN 3-540-17758-2

The term "classical groups" was coined by Hermann Weyl and used in the title of his famous book [5]. It refers to the general linear group GL_n (the group of automorphisms of an n-dimensional vector space); its unimodular (i.e., determinant 1) subgroup; the subgroups of these groups defined by the invariance of a bilinear or hermitian form, viz., the orthogonal, symplectic and unitary groups; and the projective versions of all of these groups. The importance of these groups stems from their many occurrences throughout mathematics and physics. Most frequently the base field is the real or complex field, but in the applications, for example to number theory, other fields and even division rings and more general rings have to be allowed. Since Weyl was interested in the representation theory and, inseparably, the invariant theory of his groups, the focus on the groups in his book is mainly an external one. On the other hand, the focus on the groups in [4], a later book with a similar title, is an internal one. Can one find a handful of simple elements that generate these groups and how they interact in doing so? Can one determine the structure of these groups: are they simple and, if not, what are their normal subgroups? What are the automorphisms of these groups and the isomorphisms among them? These are the main questions considered by Dieudonné, for groups over fields or division rings, and he gave an account of the answers, as they were known at that time. Since then many new results have been obtained in this area, including many by the current authors. In writing their book, it is their purpose to bring the situation up to date, and this they do very well with a comprehensive account of the subject. To keep the book at a manageable size (576 pages), they have to omit some of the proofs of the big theorems, but many are included; and in any case the references to the literature are substantial and complete. The authors follow the case-by-case format of Dieudonné in which the various types of groups are treated separately: first the general linear groups and their relatives, then the orthogonal groups, then .... There is another approach in which the different types of groups can be treated simultaneously, and the exceptional groups as well, via the theory of Lie groups and its algebraic version, the theory of algebraic groups. This is carried out in [3] where many of Dieudonné's results, at least those regarding automorphisms and isomorphisms, are subsumed, and also in [2] where analogous results for discrete subgroups of Lie groups are obtained. The current authors are, of course, well aware of this. The advantage of their case-by-case approach is that it enables them to consider more general base rings and subgroups and to go deeply into the special features that the different types of groups have.

We continue with a brief account of the contents of the book. In Chapter 1, general properties of $GL_n(R)$ (and $PGL_n(R)$), with R a quite general ring, are considered, and some of the main players are introduced: the transvections. An element σ of GL_n is a transvection if the kernel of $1-\sigma$ is a hyperplane H, the image is a line L, and the former includes the latter. This assumes that R is a field. If R is an arbitrary ring, the definition is a bit more complicated. The identity matrix with some off-diagonal entry replaced by a nonzero number is the simplest example of a transvection. These simple transvections generate a substantial part of the group and satisfy certain well-known "elementary relations" regarding their commutators. In Chapter 2, R is a field or a division ring. The simplicity of $SL_n(R)$ modulo its center is proved, and generators (the simple transvections above) and relations are given. Here algebraic K-theory comes into the picture, since $K_2(R)$ is, by definition, the kernel of the projection onto $SL_n(R)$ of the abstract group generated by the simple transvections subject to the elementary relations and hence it supplies the extra relations needed to define $SL_n(R)$ completely. In Chapter 3, the isomorphisms among the various groups are found, assuming that $n \geq 3$. In fact, this is done for many subgroups, those that are full. A subgroup G is called full if for each hyperplane H and each line L contained in it there is a transvection σ in G such that H and L are, respectively, the kernel and image of $1-\sigma$. For example, the whole group $GL_n(\mathbf{Q})$ is full (if $R = \mathbf{Q}$), certainly, but so is $SL_n(\mathbf{Z})$ and any group commensurate with it, i.e., so is any arithmetic subgroup. The following theorem is proved. Let G and G' be full subgroups of $PGL_n(D)$ and $PGL_{n'}(D')$ with n and $n' \geq 3$ and D and D' division rings. Let $\theta\colon G \to G'$ be an isomorphism. Then θ is implemented by a semilinear isomorphism of D^n onto $D'^{n'}$ (a linear isomorphism of D^n onto $D^{n'}$ composed with an isomorphism of D onto D'), or else by a semilinear anti-isomorphism (in which multiplication in D is reversed) composed with the inverse transpose. Since n' and D' may equal n and D in the assumptions, this theorem also describes all of the automorphisms of each of the groups G above. In Chapter 4, normal subgroups are considered. If I is any ideal of R, the kernel of reduction $\operatorname{mod} I$ is a normal subgroup, $SL_n(R, I)$, of $SL_n(R)$. A somewhat larger normal subgroup, $SL'_n(R)$, consists of the elements that are central scalars $\operatorname{mod} I$. Any subgroup in between these two is also normal. The congruence subgroup theorem, when it holds, states that these are the only normal subgroups. The theorem of Chapter 2 above, for example, states that it holds when R is a division ring since then every ideal is either 0 or R while every normal subgroup is either central or the whole group. It also holds when $n \geq 3$ and R is the ring of integers of an algebraic number field, except when the field is totally imaginary, in which case a measure of the amount by which it fails can be given. In the first step of the proof, one shows that if N is any normal subgroup then there is a unique ideal I such that $E_n(R, I) \subseteq N \subseteq SL'_n(R, I)$, with $E_n(R, I)$ denoting the normal subgroup generated by the transvections that are 1 mod I. Here R and n have to satisfy certain "stable range conditions" which come from algebraic K-theory (and are fully developed in the text). For $R = \mathbf{Z}$, for example, $n \geq 3$ will suffice. The second step, usually a much harder one, is to show that $E_n(R, I) = SL_n(R, I)$, or else to determine the structure of the quotient. For number rings, for example, very serious number theory is involved here (see [1]). Very wisely, the authors present the first step in full detail and then settle, mainly, for a compendium of results for the second. In the last five chapters the kind of development just sketched is carried out for the orthogonal, symplectic, and unitary groups, but the situation is now more complicated, at the start and therefore throughout the development. And the authors have to put conditions on the bilinear and hermitian forms allowed to ensure the existence of enough transvections, or elements like them, to carry out their program. In this respect it should be mentioned that Dieudonné, Borel, and Tits in [2, 3, 4] had to do likewise.

Have the authors succeeded in their goal of giving "an account of the fundamental algebraic properties of the classical groups over rings"? They certainly have, with a very comprehensive treatment which will be the standard reference for some time to come. The bibliography is more than adequate, but the index, in my opinion, is a bit skimpy: I found myself on occasion thumbing the pages looking for items that I had seen earlier. But this is a minor quibble. The authors should be thanked by the mathematical community for their excellent book and for the hard work that they put into it.

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Harmonic analysis of spherical functions on real reductive groups, by R. Gangolli and V. S. Varadarajan. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 101, Springer-Verlag, Berlin, Heidelberg, and New York, 1988, xiv+365 pp., $110.00. ISBN 3-540-18302-7

1

Let $\mathscr{X}$ be a locally compact Hausdorff space endowed with a transitive action of a locally compact group G. Then $\mathscr{X} = G/K$ for a closed subgroup K. If $\mathscr{X}$ also admits an invariant measure, then G acts on $L^2(\mathscr{X})$ by unitary transformations by the formula

$$L_{\mathscr{X}}(g)f(x) = f(g^{-1}x). \tag{1.1}$$

The study of the decomposition of this representation into a "direct integral" of irreducible components is usually known as harmonic analysis on homogeneous spaces.

Assume that $\mathscr{X} = G/K$ is Riemannian symmetric. A special role is played by $C_c(G//K)$, the space of continuous compactly supported functions on G which are K-invariant under the regular representation $(g_1, g_2) \cdot f(x) = f(g_1^{-1}xg_2)$. Gel'fand [Ge], observed that under convolution, $L^1(G//K)$ is an abelian Banach