the complex Monge-Ampère operator. By first studying general capacity set functions, the author focuses attention on the analytic problems that arise in proving important properties of the capacity, such as continuity under decreasing limits of compact sets. The continuity of $(dd^c u)^n$ under bounded, monotone limits of plurisubharmonic functions is proved, as is the equivalence of negligible sets (with respect to plurisubharmonic functions), pluripolar sets, and sets of capacity zero. Other interesting applications are given to the study of the (pluri-) Green function and Siciak's global extremal function, the analogue of the Green function with pole at infinity in one complex variable.

The major shortcoming of the book is that it does not supply any outline or overview of the subject. There should have been some introductory material in each chapter that calls attention to the main results and the direction one takes to prove them. Also, I did not find any strong connection between the last three chapters and the topics discussed in the first nine chapters. A surprising omission in a book on capacities in several complex variables is that there is no mention of some of the most interesting and important new capacities, such as the projective capacities studied by Sibony and Wong and by H. Alexander, and the capacity associated with the "transfinite diameter," studied by Siciak and Zaharjuta. Of course, it is impossible to have everyone's favorite topics in such a short monograph.

Since these notes are lecture notes from courses given by the author, it is perhaps not surprising that there are many typographical errors in them. However, I found no serious errors. All in all, I think this book is a good source for obtaining an introduction to a new and interesting subject.

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Computability in analysis and physics, by M. B. Pour-El and J. I. Richards. Perspect. Math. Logic, Springer-Verlag, New York, Berlin, Heidelberg, 206 pp. ISBN 0-387-50035-9

Which processes in analysis and physics preserve computability, and which do not? In order to answer this question, after an outline of the necessary minimum of recursion theory, the authors introduce concepts of effective convergence, computable sequence of real numbers, and computable sequence of functions.

Let X be a Banach space, (x_{nk}) a double sequence in X, and (x_n) a sequence in X. We say that $x_{nk} \to x_n$ *effectively in* k *and* n *as* $k \to \infty$ if there exists a recursive function $e: \mathbf{N}\times\mathbf{N} \to \mathbf{N}$ such that $\|x_{nk} - x_n\| \leq 2^{-N}$ for all n and N, and all $k \geq e(n, N)$.

A *computable sequence* (r_n) *of rational numbers* is characterized by three recursive functions a, b, and s from $\mathbf{N}$ to $\mathbf{N}$ such that for all n, $b(n) \neq 0$ and $r_n = (-1)^{s(n)}a(n)/b(n)$; computable double sequences are defined here, and later in an abstract setting, in terms of computable sequences, using standard pairing bijections from $\mathbf{N} \times \mathbf{N}$ onto $\mathbf{N}$. A *computable sequence* (x_n) *of real numbers* is defined by a computable double sequence (r_{nk}) of rational numbers such that $r_{nk} \to x_n$ effectively in k and n as $k \to \infty$. A *computable real number*, or *recursive real number*, is a real number x such that the sequence $(x, x, \ldots)$ is computable; the terms of a computable sequence of real numbers are, of course, computable as real numbers. Corresponding notions of computability involving complex numbers are introduced in the obvious way, using real and imaginary parts.

It is a simple consequence of the effective enumerability of the set of all partial recursive functions on $\mathbf{N}$ that the set of all computable real numbers is countable [Ro, Chapter 1, Theorem I]; whence there exist uncountably many noncomputable real numbers.

A *computable sequence of real functions* is a sequence (f_n) of functions from $\mathbf{R}$ to $\mathbf{R}$ such that:

> For each computable sequence (x_k) of real numbers, the double sequence $(f_n(x_k))$ of real numbers is computable; and
>
> There exists a nonvanishing recursive function $d: \mathbf{N}^3 \to \mathbf{N}$ such that for all m, n, and N, and all $x, y \in [-m, m]$,
>
> $$|x - y| \leq 1/d(m, n, N) \Rightarrow |f_n(x) - f_n(y)| \leq 2^{-N}.$$

A similar definition applies to the notion of a computable sequence of functions $f_n: I \to \mathbf{R}$, where I is a compact interval with recursive endpoints. If I is such an interval or $\mathbf{R}$ itself, then a *computable real-valued function* on I is a function $f: I \to \mathbf{R}$ such that the sequence $(f, f, \ldots)$ is computable. Such a function f is *effectively uniformly continuous* on each compact subinterval J of I with recursive endpoints, in the sense that there exists a nonvanishing recursive function $d: \mathbf{N} \to \mathbf{N}$ such that for all x, y in J and all N,

$$|x - y| \leq 1/d(N) \Rightarrow |f(x) - f(y)| \leq 2^{-N}.$$

If (f_n) is a computable sequence of functions on I, then each f_n is a computable function.

Since the appearance of Turing's seminal paper in 1936 [Tu], and the subsequent development of the notions of computability described above, a number of interesting results have been obtained in the theory of computability in elementary analysis. Among these are:

Specker's theorem. *There exists a strictly increasing computable sequence (r_n) of rational numbers in* $[0, 1]$ *that is eventually bounded away from any recursive real number* [Sp; BR, Chapter 3, (3.1)],

and

Myhill's theorem. *There exists a computable function $f: [0, 1] \to \mathbf{R}$ that is C^1 and twice differentiable, whose derivative f' is not a computable function.*

A restricted version of the latter was first proved in [My].

Where Pour-El and Richards have broken new ground is in the theory of computability in Banach spaces, which they approach axiomatically using the notion of a *computability structure*: that is, a pair $(X, \mathcal{S})$ consisting of a Banach space X and a nonempty set $\mathcal{S}$ of sequences in X, satisfying the following axioms.

Axiom 1. *If (x_n) and (y_n) are in $\mathcal{S}$, (α_{nk}) and (β_{nk}) are computable double sequences of real or complex numbers, and $d: \mathbf{N} \to \mathbf{N}$ is a total recursive function, then the sequence with* nth *term*

$$\sum_{k=0}^{d(n)} (\alpha_{nk} x_k + \beta_{nk} y_k)$$

is in $\mathcal{S}$.

Axiom 2. *If (x_{nk}) is a double sequence in $\mathcal{S}$ such that $x_{nk} \to x_n$ effectively in k and n as $k \to \infty$, then $(x_n) \in \mathcal{S}$.*

Axiom 3. *If* $(x_n) \in \mathcal{S}$, *then* $(\|x_n\|)$ *is a computable sequence of real numbers.*

The elements of $\mathcal{S}$ are called the *computable sequences of the structure*, and $\mathcal{S}$ is also described as a *computability structure for the Banach space* X.

It follows easily from the axioms that the sequence $(0, 0, 0, \ldots)$ belongs to $\mathcal{S}$, that a recursively indexed subsequence of an element of $\mathcal{S}$ is in $\mathcal{S}$, and that the sequence obtained by interlacing two elements of $\mathcal{S}$ is in $\mathcal{S}$.

To each of the standard Banach spaces that are studied in analysis, there corresponds a natural intrinsic computability structure. For example, if a and b are computable real numbers, then the intrinsic computability structure for the Banach space $C[a, b]$, with the uniform norm, consists of all sequences of functions from $[a, b]$ to $\mathbf{R}$ that are computable according to the elementary definition given earlier. For another example, a sequence (f_n) of functions in $L^p(\mathbf{R})$ $(p \neq \infty)$ is L^p*-computable* if there exists a computable double sequence (g_{nk}) of continuous functions from $\mathbf{R}$ to $\mathbf{R}$ such that the support of g_{nk} is contained in $[-k, k]$, and such that $\|f_n - g_{nk}\|_p \to 0$ effectively in k and n as $k \to \infty$; the set of L^p-computable functions constitutes the intrinsic computability structure on $L^p(\mathbf{R})$.

A computability structure $(X, \mathcal{S})$ is *effectively separable* if there exists an element (e_n) of $\mathcal{S}$—called an *effective generating set* for $(X, \mathcal{S})$ or, loosely, for X—whose linear span is dense in X. The monomials $1, x, x^2, \ldots,$ and the set of all step functions with rational values and jump points, form effective generating sets for the intrinsic computability structures on $C[0, 1]$ and $L^p(\mathbf{R})$, respectively.

The definition of effective generating set does not require that the linear span of (e_n) be effectively dense in $\mathcal{S}$, in any natural sense; but the effective density of (e_n) is a fundamental consequence—the **effective density lemma**—of the authors' definition of computability structure. The effective density lemma has some interesting applications. For example, by applying it to $C[0, 1]$ and the effective generating set $\{1, x, x^2, \ldots\}$, we immediately obtain an effective version of the Weierstrass approximation theorem; and by applying it the information about $L^1(\mathbf{R})$ provided by the classical Wiener Tauberian theorem, we obtain an effective version of that theorem. In a slightly different vein, the effective density lemma enables us to show that if two computability structures on X have a common effective generating set, then they have the same set of computable sequences (the **stability lemma**).

The material we have discussed so far belongs to the second of the three parts in which this book is divided; the first part comprises two preliminary chapters that cover computability in analysis from first principles—that is, in a nonaxiomatic fashion. The remainder of the second part deals with the first of the authors' two main theorems and their consequences.

The **first main theorem** establishes a link between boundedness and the preservation of computability for closed linear operators between Banach spaces:

> *Let T be a closed linear mapping between Banach spaces X and Y with computability structures, and suppose there exists an effective generating set (e_n) for the computability structure on X, such that the sequence (Te_n) is computable in Y. Then T maps computable elements of its domain to computable elements of Y if and only if T is bounded.*

The main part of the proof actually provides an algorithm which, applied to an unbounded operator T satisfying the hypotheses, constructs a computable element x of the domain of T such that Tx is not a computable element of Y.

A corollary of the first main theorem is that a bounded closed linear mapping $T: X \to Y$ (whose domain is the entire Banach space X) maps computable sequences in X to computable sequences in the Banach space Y.

The authors give many interesting applications of their first main theorem, among which are simple proofs of results proved by more elementary and intricate methods in earlier chapters. Of particular interest is their discussion of noncomputable solutions of the three-dimensional wave equation

$$(1) \qquad \nabla^2 u = u_{tt}, \qquad u(\mathbf{x}, 0) = f(x), \qquad u_t(\mathbf{x}, 0) = 0,$$

where $\mathbf{x}$ ranges over the cube $[-1, 1]^3 \subset \mathbf{R}^3$, $0 \le t < 2$, and the initial function f is defined and continuous on the cube $[-3, 3]^3 \subset \mathbf{R}^3$. The classical solution of (1), under these physically meaningful conditions on the domains involved, is given by *Kirchhoff's* *formula* [cf. Pe, page 80]

$$u(\mathbf{x}, t) = \iint_{\text{unit sphere}} [f(\mathbf{x} + t\mathbf{n}) + t\nabla f(\mathbf{x} + t\mathbf{n}) \cdot n] dS(\mathbf{n}),$$

where $dS(\mathbf{n})$ is the area measure on the unit sphere, normalized so that the total area of the sphere is 1. The authors show that for fixed t, the unbounded operator

$$f \mapsto \iint_{\text{unit sphere}} [f(\cdot + t\mathbf{n}) + t\nabla f(\cdot + t\mathbf{n}) \cdot n] dS(\mathbf{n})$$

is closed, and maps monomials in x, y and z to computable elements of the range of T; since those monomials form an effective generating set for the computability structure under consideration, it follows immediately from the first main theorem that there exists a computable continuous function f such that the solution $u(\mathbf{x}, 1)$ of (1) at time $t = 1$ is a continuous, but not computable, function of $\mathbf{x}$.

This result is a good illustration of the power of the first main theorem, made available by the authors' very general notion of a computability structure. Further evidence of the advantage of this generality is revealed in the same context: The authors show that, relative to a natural computability structure associated with the *energy norm* $\|u(\mathbf{x}, t)\| \equiv \sup_t |E(u, t)|$, where

$$E(u, t)^2 \equiv \iiint_{\mathbf{R}^3} [|\nabla u|^2 + (\partial u/\partial t)^2] d\mathbf{x},$$

the solution of the wave equation (1) with computable initial data *is* computable on $\mathbf{R}^3 \times [-M, M]$, where M is any positive recursive real number.

The third and final part of the book deals with the computability theory of eigenvalues and eigenvectors for a closed operator $T: H \to H$, where H is an effectively separable Hilbert space and T is *effectively determined*: That is, for some computable sequence (e_n) in H, the ordered pairs $(e_n, Te_n)(n = 1, 2 \ldots)$ form an effective generating set for the graph of T relative to its natural computability structure; in which case the sequence (e_n) is an effective generating set for H. The authors' fundamental result on eigenvalue theory is their **second main theorem,** of which the following is the more interesting part:

> *Let T be an effectively determined self-adjoint operator on an effectively separable Hilbert space H. Then there exist a computable sequence (λ_n) of real numbers, and a recursively enumerable*[1] *set $A \subset \mathbf{N}$ such that*
>
> (i) *the spectrum of T is the closure of $\{\lambda_n : \in \mathbf{N}\}$; and*
>
> (ii) *the set of eigenvalues of T is $\{\lambda_n : n \in \mathbf{N} \backslash A\}$.*

The proof of this theorem occupies most of the 40-odd pages in the final chapter of the book, and includes algorithms, with proofs of their correctness, for the computation of the sequence (λ_n) and the effective enumeration of the recursively enumerable set A. Rather than dwell on that proof, I prefer to describe some of the interesting results associated with the theorem itself.

First, although, by the theorem, the individual eigenvalues of an effectively determined self-adjoint operator T on an effectively separable Hilbert space H are computable, the *sequence* of eigenvalues need not be computable, even if T is bounded; but if T is compact, then the sequence of eigenvalues is computable. Secondly, there exists an effectively determined bounded operator T on $L^2[0, 1]$, one of whose eigenvalues is a noncomputable real number. Thirdly, the **eigenvector theorem** asserts that there exists an effectively determined compact self-adjoint operator on $L^2[0, 1]$ for which 0 is an eigenvalue of multiplicity 1 and none of the corresponding eigenvectors is computable. In each of the last two results, $L^2[0, 1]$ is taken with its intrinsic computability structure.

The proof of the eigenvector theorem is of especial interest: The authors first establish the result relative to an *ad hoc* effectively separable computability structure on $L^2[0, 1]$; they then prove that any two effectively separable computability structures on a Hilbert space are isomorphic, and use this isomorphism to complete the proof of the full form of the eigenvector theorem. To illustrate the dependence of their arguments on the context of a separable Hilbert space, the authors also give examples of nonisomorphic effectively separable computability structures on the Banach space l^1.

---

[1] A subset A of $\mathbf{N}$ is *recursively enumerable* if either $A = \varnothing$ or there is a computable function from $\mathbf{N}$ onto A (that is, an effective listing of the elements of A).

Having given some idea of the flavor of the book, I now turn to the following remark in the authors' "Addendum: Open Problems."

> ⋯ the reasoning in this book is classical—i.e. the reasoning used in everyday mathematical research. This contrasts with the intuitionist approach (e.g. of Brouwer), the constructivist approach (e.g. of Bishop), and the Russian school (e.g. Markov and Shanin). A natural question is: What are the analogs, within these various modes of reasoning, of the results in this book?

The distinction between classical and intuitionistic (constructive) logic is significant in several places in recursive mathematics. To illustrate this, consider the proposition:

(2) *Every recursive real number has a recursive binary expansion.*

(A *recursive binary expansion* of a recursive real number x is a total recursive function $f_x:\mathbf{N} \to \{0, 1\}$ such that $x = \sum_{n=0}^{\infty} f_x(n)2^{-n}$.) One proof of (2) proceeds like this. Either $x = p/q$ for some integers p and q, in which case we obtain the nth place of a recursive binary expansion of x by performing, in binary arithmetic, as much as is necessary of the long division of p by q; or else x is irrational. In the latter case, there is a recursive procedure which, applied to any rational number r, enables us to decide whether $x < r$ or $r < x$; using this procedure, and an interval-halving argument applied first to an interval containing x and having rational endpoints, we can construct a recursive binary expansion of x.

This proof embodies two distinct algorithms, one applicable when x is rational, the other when x is irrational. Within the framework of classical logic, it is an acceptable proof of the statement we set out to prove: It actually proves the statement in the form:

> *For each recursive real number x, there exists an algorithm which produces a recursive binary expansion of x.*

However, the split into cases used in our proof is not acceptable within intuitionistic logic unless we have a procedure for deciding, for any given recursive real number x, which of the two cases obtains; such a decision procedure would then be adjoined at the start of the above proof to convert it into a fully constructive one. Unfortunately, there is no algorithm for deciding whether or not a given recursive real number is rational, so there is no possibility of converting the above proof into a constructive one. Of course, there remains the possibility that there is a totally different proof of (2) that is constructive. But this is not the case: It is not hard to show that a constructive proof of (2) would convert into one of the following proposition, which is false even in classical recursion theory [BR, Chapter 3, (1.6)]:

> *There is an algorithm which, applied to a binary sequence* (a_n) *with at most one term equal to* 1, *outputs* 0 *if* $a_n = 0$ *for all even* n, *and* 1 *if* $a_n = 0$ *for all odd* n.

Another way of looking at the constructive problem with (2) is to note that in the classical proof sketched above, the algorithm for computing the recursive binary expansion of x depends on x itself: It is not *uniform* in x. As a rule (which, like all such rules, should not be taken as infallible), if classical logic enables us to prove that a single algorithm provides certain information about all recursive real numbers, then intuitionistic logic will do the same, perhaps with some modifications to the input or the structure of the actual algorithm.

For example, if a is a positive recursive real number, then there is an algorithm which enables us to decide, for any given recursive real number x and using classical logic, whether $a > x$ or $x > 0$. Here is a high-level description of the algorithm:

> Choose a sequence (r_n) of rational numbers, and a recursive function $e: \mathbf{N} \to \mathbf{N}$, such that $|a - r_n| \leq 2^{-N}$ whenever $n \geq e(N)$.
>
> Compute $e(N)$ and $r_{e(N)}$ for $N = 0, 1, 2, \ldots$, until we obtain a value N such that $r_{e(N+2)} > 2^{-N}$.
> Given any computable real number x, compute a rational number s such that $|x - s| < 2^{-N-2}$.
>
> Decide whether $s \leq 2^{-N-1}$ or $s > 2^{-N-1}$; in the first case, $a > x$; in the second, $x > 0$.

For the application of this classical algorithm, we only need the information that $a \leq 0$ is impossible. Using intuitionistic logic, we can show that the same algorithm does the same job, provided that the information "$a > 0$" is given in a positive form: for example, a rational number r and a positive integer N such that $|a - r| < 2^{-N-2}$ and $r > 2^{-N}$.

Some people would argue that even constructively this information is already embodied in the statement "it is impossible that $a \leq 0$." The basis of this argument is their acceptance of

Markov's principle. *If $P(n)$ is a decidable property of natural numbers n, such that $\neg\forall n \neg P(n)$, then there exists a value ν such that $P(\nu)$,*

which represents a form of unbounded search. The general view, however, is that Markov's principle is at best of doubtful significance within the framework of intuitionistic logic.

In connection with the authors' second main theorem, we have already observed that, classically, the eigenvalues of an effectively determined self-adjoint operator T on an effectively separable Hilbert space H are computable real numbers but need not form a computable sequence. In other words, for each $n \in \mathbf{N}$ there exists an algorithm, depending on n, which computes the nth, eigenvalue in an enumeration of the set of all eigenvalues of T; but, for a certain T, there is no *uniform* algorithm that applies to all natural numbers n and computes, from the input n, the nth eigenvalue in such an enumeration. This can be shown also by modifying the example on pages 20–21 of [Br], to demonstrate constructively the impossibility of a uniform recursive algorithm for the computation of the eigenvalues of positive compact self-adjoint operators on a two-dimensional Hilbert space. It appears, therefore, that the authors' second main theorem, and perhaps much of their work on eigenvalues and eigenvectors, will not readily translate into a constructive form.

If we turn back to the proof of their first main theorem, we find that there, too, the constructive interpretation is elusive. In this case, the problem is that if an operator $T: X \to Y$ between normed spaces is defined constructively, then the elements of its domain and range are, of necessity, computable objects; so we cannot have the situation where x belongs to the domain of T and Tx is not computable. Inspection shows that with the help of Markov's principle we can extract from the authors' arguments a proof of a cognate constructive theorem; but this theorem seems to have little intrinsic merit and no interesting applications like those of its classical counterpart. So the question remains: What, if any, are the significant analogues, within constructive mathematics, of the first main theorem?

Now, constructive mathematics is not solely a matter of restriction to recursive objects and intuitionistic logic, although *formally* that is what distinguishes the recursive constructive mathematics (RUSS) of the Russian School from classical mathematics [BR, Chapters 1 and 3; Ku]. The variety of constructive mathematics (BISH) advocated by the late Errett Bishop uses intuitionistic modes of reasoning, is based on a primitive notion of algorithm, and does not confine its attention to recursive objects [Bi, BB]. In consequence, *every theorem of* BISH, *suitably interpreted, is also a theorem of* RUSS *and of classical mathematics*; but theorems of constructive mathematics that depend on recursive function theory are unobtainable within BISH.

With these comments in mind, I would like to discuss further the work of Pour-El and Richards, in [PR] and the book under review, on the wave equation. When considered within RUSS, the construction and uniform continuity of the function f that they use in [PR] to provide initial conditions leading to a noncomputable solution of (1) create no problems; but the argument they use to prove that f is differentiable—in fact, of class C^1—depends on the term-by-term differentiability of a series of functions under conditions that are not sufficient to ensure the applicability of the relevant constructive theorem [BB, Chapter 2, (6.10)]. Taken with our uncertainty over significant constructive analogues of the first main theorem, this raises serious doubts about a constructive counterpart of the authors' nonrecursive solution of (1) with recursive initial data. Of course, any such counterpart would be expressed in terms of the *nonexistence* of a solution of (1) at time $t = 1$, since noncomputable objects cannot be perceived within a rigidly constructive framework.

Another point to note is that if a computable function f leads classically to a noncomputable solution u of (1) on the compact domains in question, then u is not effectively uniformly continuous [PR, page 237, Proposition]; in particular, the solution is *weak*—that is, not of class C^2.

Contrast this with what happens in BISH: In that framework, if the function f is uniformly and continuously differentiable on the appropriate compact domain in $\mathbf{R}^3$, then Kirchhoff's formula gives the unique solution to the wave equation (1), and this solution is of class C^2. Also, if the wave equation has a solution u on a compact set, then we expect to be able to prove that u is uniformly continuous, since if it were not, we would have a contradiction to the classical uniform continuity theorem.

Incidentally, since weak solutions of the wave equation are physically significant, we should attempt to accommodate them within any constructive theory intended to reflect reality. One way to do this within BISH might be to enlarge the class of admissible initial functions for (1) to include every function f that is uniformly continuous on compact subsets of its domain $D \subset \mathbf{R}^3$, and whose gradient is defined almost everywhere, and Lebesgue integrable, on D (cf. [Bi, Chapter 8, Theorem 5]).

The results of Pour-El and Richards in connection with (1) seem to suggest that in spite of the intuitive connection between computability and predictability in physics, noncomputable numbers and functions necessarily arise even in basic areas like the theory of waves. But we do not know whether the intuitive connection fails to hold in practice—that is, whether there is any *physically realizable* system represented by the data (1) with recursive initial conditions but nonrecursive solutions (cf. [Kr]). Of course, if we follow Bishop and work within a rigorously constructive framework, eschewing Church's thesis as a general principle, then the problem of noncomputable solutions of the wave equation does not arise.

To summarise: the book under review provides an intriguing introduction to the authors' research on the interplay between recursion theory and analysis. The presentation is clear and careful, aimed perhaps more at recursion theorists with a limited background in analysis than at analysts. Nonetheless, an analyst—indeed, any mathematician—with an interest in questions of computability will enjoy reading this book, and will find therein an abundance of directions in which to continue the pioneering research of its authors.

REFERENCES

[Bi] E. Bishop, *Foundations of constructive analysis*, McGraw-Hill, New York, 1967.

[BB] D. Bridges and E. Bishop, *Constructive analysis*, Grundlehren Math. Wiss., vol. 279, Springer-Verlag, Berlin, 1985.

[BR] D. Bridges and F. Richman, *Varieties of constructive mathematics*, London Math. Soc. Lecture Note Ser., vol. 97, Cambridge Univ. Press, London, 1987.

[Br] D. Bridges, *A constructive look at positive linear functionals on* $L(H)$, Pacific J. Math. **95** (1981), 11–25.

[Kr] G. Kreisel, *A notion of mechanistic theory*, Synthese **29** (1974), 11–26.

[Ku] B. A. Kushner, *Lectures on constructive mathematical analysis*, Amer. Math. Soc., Providence, R.I., 1985.

[My] J. Myhill, *A recursive function, defined on a compact interval and having a continuous derivative that is not recursive*, Michigan Math. J. **18** (1971), 97–98.

[Pe] I. G. Petrovskii, *Partial differential equations*, Wiley, Philadelphia, 1967.

[PR] M. B. Pour-El and I. Richards, *The wave equation with computable initial data such that its unique solution is not computable*, Adv. in Math. **39** (1981), 215–239.

[Ro] H. Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967.

[Sp] E. Specker, *Nicht konstruktiv beweisbare Sätze der Analysis*, J. Symbolic Logic **14** (1949), 145–158.

[Tu] A. M. Turing, *On computable numbers, with an application to the "Entscheidungsproblem"*, Proc. London Math. Soc. (Ser. 2) **42** (1936), 230–265; corr. ibid. **43** (1937), 544–546.

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The Riemann problem and interaction of waves in gas dynamics, by Tung Chang (Tong Zhang) and Ling Hsiao (Ling Xiao). Longman Scientific and Technical (Pitman Monographs No. 41), Essex, 1989, 272 pp. ISBN 0-582-01378-X

Many phenomena involving nonlinear wave motion fit into the mathematical framework of the so-called "hyperbolic systems of conservation laws." These are systems of nonlinear partial differential equations which describe the conservation of certain physical quantities, e.g., mass, momentum, energy, etc. The equations take the form $\operatorname{div} \phi(u) = 0$, where the divergence is with respect to the space-time independent variables, and ϕ is a nonlinear function of the unknown state variable u.

The most mathematically well-understood case is that of one