A COMPLETE SOLUTION TO THE POLYNOMIAL 3-PRIMES PROBLEM

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I. Introduction

By the "classical 3-primes problem" we mean: can every odd number \( \geq 7 \) be written as a sum of three prime numbers? This problem was attacked with spectacular success by Hardy and Littlewood [8] in 1923. Using their famous Circle Method and assuming the Generalized Riemann Hypothesis, they proved that there exists a positive number \( N \) such that every odd integer \( n \geq N \) is a sum of three primes. In 1937, Vinogradov [12] employed his ingenious methods for estimating exponential sums to prove the Hardy-Littlewood conclusion without invoking the Riemann Hypothesis. The result is therefore known as Vinogradov's Theorem. Vinogradov's proof actually implies a computable value for \( N \), raising the possibility that the classical 3-primes problem can be completely settled by computation. For example, by carefully estimating the errors in Vinogradov's proof, Borodzkin [2] showed that one can take

\[ N = 3^{315} \].

Unfortunately, this value is far beyond the minimum that would make the problem accessible to even the fastest computers.

If instead of \( \mathbb{Z} \) we consider the ring \( \mathbb{F}_q[x] \) of polynomials in a single variable \( x \) over the finite field \( \mathbb{F}_q \) of \( q \) elements, we can easily formulate, in direct analogy to the classical 3-primes problem, a polynomial 3-primes problem. To this end we observe that the analog of prime number is irreducible polynomial, of positive number is monic polynomial, and we need also:

Definition. A monic polynomial \( M \) over \( \mathbb{F}_q \) is called even if \( q = 2 \) and if \( M \) is divisible by \( x \) or \( x + 1 \); otherwise \( M \) is called odd (so, for all \( q \neq 2 \), all \( M \) are odd).
It is easy to show that there exist even monic polynomials of arbitrarily high degree which cannot be written as a sum of three monic irreducibles [5]. Moreover, just as 1, 3, and 5 in the classical setting are "too small" to have the desired representation, so in the polynomial setting are all linear polynomials (over all finite fields) and quadratic polynomials of the form $x^2 + \alpha$ over even finite fields "too small" to have the desired representation [5]. Thus we must omit these cases from consideration.

**Definition.** A monic polynomial $M$ over $F_q$ of degree $r$ is said to be a 3-primes polynomial if it can be written as a sum of three irreducible monic polynomials over $F_q$, one of degree $r$ and the other two of lesser degree.

The following theorem provides a complete solution to the polynomial 3-primes problem:

**The Polynomial 3-Primes Theorem.** Every odd monic polynomial $M$ of degree $r \geq 2$ over every finite field $F_q$ (except the case $M = x^2 + \alpha$ with $q$ even) is a 3-primes polynomial.

The proof of this theorem falls naturally into three parts:

1. An Asymptotic Theorem analogous to Vinogradov's Theorem in the classical setting.
2. Subtheorems which reduce the cases not covered by the Asymptotic Theorem to a finite, tractable number.
3. A computer check of all remaining cases.

In the remainder of this announcement, we summarize these three parts.

**II. THE ASYMPTOTIC THEOREM**

A complete exposition of the proof of the following theorem is contained in [7]. See also [3] and [10].

**Asymptotic Theorem.** For every degree $r \geq 5$ there exists a $q_r$, depending on $r$ and decreasing as $r$ increases, such that if $q \geq q_r$, then every odd monic polynomial of degree $r$ over $F_q$ is a 3-primes polynomial. Moreover, we have $q_r = 2$ for all sufficiently large $r$.

The method of proof is the Hardy-Littlewood Circle Method adapted to the function field setting. The analog for the unit circle $T$ is the adéle class group $C_k = A_k/k$ with $k = F_q(x)$ (cf. [11]).
The normalized Haar measure $dt$ on the compact $k$-vector space $C_k$ is a natural replacement for the complex path integral around $T$. After the choice of the generator $x$ of $k$, there is a canonical additive character $E: A_k \rightarrow T$ which is defined as follows

$$E(t) = e_q(\text{res}(t\,dx)) \quad \text{for} \quad t \in A_k,$$

where $e_q$ is the usual additive character on $F_q$. By the residue theorem, the discrete subgroup $k$ of $A_k$ lies in the kernel of $E$, and so $E$ can be regarded as a character on $C_k$.

For $t \in A_k$, we introduce the functions

$$F_r(t) = \sum_{\deg P = r} E(Pt) \quad \text{and} \quad H_r(t) = \sum_{\deg P < r} E(Pt)$$

and observe in the familiar way that $F_r(t) \cdot H_r^2(t)$ is a generating function for the number of representations $N(M)$ of the monic polynomial $M$ as a 3-primes polynomial. Therefore

$$N(M) = \int_D F_r(t)H_r^2(t)E(-Mt)\,dt$$

where $D \subset A_k$ is any fundamental domain for $C_k$. It remains to estimate $F_r(t)$ by simpler functions and to choose $D$ so that the error term is as small as possible. In estimating $F_r(t)$, one can imitate the original Hardy-Littlewood line of attack because the analog of the Generalized Riemann Hypothesis is a consequence of Weil's celebrated proof of the Riemann Hypothesis for smooth projective curves over $F_q$. The resulting approximation to $F_r(t)$ is good when the denominator

$$\partial(t) = \prod_{P} P^{\max\{0, -v_P(t_P)\}}$$

of the adèle $t$ satisfies

$$\deg \partial(t) \leq r/2 \quad \text{and} \quad v_\infty(t_\infty) > r/2 + \deg \partial(t),$$

where $\infty$ is the infinite place of $k$. The union $D$ of all $t \in A_k$ which satisfy these relations is the analog of the Farey dissection, and this $D$ is indeed a fundamental domain for $C_k$. Just as in the Hardy-Littlewood approach to the classical 3-primes problem, "minor arcs" are not required.

The end result of the work is an asymptotic formula for $N(M)$ with a very good error term

$$N(M) = (1/r)(L_{r-1}(q))^2S(M) + O(q^{7r/4} / ((q - 1)(r - 1)))$$
where

\[ L_{r-1}(q) = \sum_{1 \leq i \leq r-1} q^i / i \]

and \( S(M) \) is the "singular series." The Asymptotic Theorem then follows from the facts that

\[ L_{r-1}(q) \geq q^r / ((q - 1)(r - 1)) \]

and that \( S(M) \) is bounded below by a strictly positive constant which is independent of \( q \).

Now it is possible to make a careful evaluation of the constant in the error term of the asymptotic formula above, obtaining for each \( r \geq 5 \) a lower bound for \( q_r \) (see [7]). The results of this evaluation are summarized in the following table. (This data is, of course, the polynomial analog of Borodzkin's astronomical \( N \).

<table>
<thead>
<tr>
<th>NUMERIC RESULTS FOR THE ASYMPTOTIC THEOREM</th>
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<tbody>
<tr>
<td>For odd monic polynomials of degree ( r = )</td>
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<td>34 - 41</td>
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<tr>
<td>42 and up</td>
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It remains then to "fill in" these remaining cases.

III. THE SUBTHEOREMS

The first subtheorem covers the low degree cases at the top of Table 1.

**Subtheorem 1.** Every odd monic polynomial \( M \) of degree \( r = 2, 3, 4, \) or 5 over every finite field \( \mathbb{F}_q \) is a 3-primes polynomial except for the case \( M = x^2 + \alpha, \) \( q \) even. Every monic polynomial
of degree \( r = 6 \) is a 3-primes polynomial provided that \( q \geq 19 \). Every monic polynomial of degree \( r = 7 \) is a 3-primes polynomial provided that \( q \geq 211 \) but \( q \neq 256 \).

See [4] and [5] for the \( q \) odd and \( q \) even cases respectively. The methods employed are primarily affine geometry over finite fields (as in Artin [1]), although the cases \( r = 6 \) and \( r = 7 \), \( q \) odd require in addition the Riemann Hypothesis for certain nonabelian Artin \( L \)-functions.

Combining Subtheorem 1 with the Asymptotic Theorem does indeed reduce the polynomial 3-primes problem to a finite calculation, but as it stands an intractable one. For example, to check the \( 3^{33} \) monic polynomials of degree 33 over \( F_3 \) at a rate of one per millisecond would require about 176 years. More mathematics is needed.

**Subtheorem 2.** If \( q \) and \( r \) are relatively prime, then it suffices to check for 3-primes representations only of polynomials with first coefficient 0 and second coefficient 0, 1, and, for \( q \) odd, some fixed quadratic nonresidue.

Again, see [4] and [5]. This result says we can replace \( q^2 \) checks by two (for \( q \) even) or three (for \( q \) odd) checks. It helps substantially for the larger \( q \)’s remaining to be checked, but not much for the smaller \( q \)’s. For these, the following result is crucial:

**Subtheorem 3.** Among monic polynomials of degree \( r \) over \( F_q \), there exist irreducible polynomials with every possible choice of first \( s \) coefficients provided that

\[
r/2 > s + \log_q(s+1).
\]

See the proof of Theorem 9.3 of [9]. This result says that given \( M \) of degree \( r \), we can find an irreducible \( P_1 \) of degree \( r \) such that \( M - P_1 \) is monic of degree not much larger than \( r/2 \). For example, in the case \( q = 3 \), \( r = 33 \), we are assured by Subtheorem 3 of the existence of a \( P_1 \) such that \( M - P_1 \) is monic of degree 19. The combination of the Asymptotic Theorem together with the three subtheorems has now reduced the problem to a tractable computation.

**IV. THE COMPUTER CHECK**

Application of all the preceding results reduces the polynomial 3-primes problem to the following: for 85 separate combinations
of \( q \) and \( r \) (for example \( q = 256, r = 5 \), \( q = 199, r = 4 \), \ldots, \( q = 2, r = 25 \), etc.), we must check that every monic polynomial (except for odd polynomials when \( q = 2 \)) with first coefficient 0 and second coefficient 0, 1, and (for odd \( q \)) a fixed quadratic non-residue is a sum of two monic irreducible polynomials. This is still a large computation requiring a powerful computer. One of us (Effinger) programmed the IBM 3090 Supercomputer at the Cornell National Supercomputing Facility to check these remaining cases. Algorithms were designed to:

1. generate lists of irreducible polynomials, and
2. check off the sums of appropriate pairs of irreducibles.

For the former both the Berlekamp factorization algorithm for \( \mathbb{F}_q[x] \) and an "extension field" algorithm were employed. For the latter extensive indexing was used. See [6] for the details of the algorithm design.

On December 19, 1989, the IBM 3090 completed the list of the 85 cases which needed to be checked. A total of 64.8 hours of central processing was needed. A complete solution to the polynomial 3-primes problem was then at hand.

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References


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