ENDS OF RIEMANNIAN MANIFOLDS
WITH NONNEGATIVE RICCI CURVATURE
OUTSIDE A COMPACT SET

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ABSTRACT. We consider complete manifolds with Ricci curvature nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number.

1. INTRODUCTION

Toponogov [T] showed that in a complete manifold of nonnegative sectional curvature, a line splits off isometrically, i.e. any nonnegatively curved $M^n$ is isometric to a Riemannian product $N^k \times R^{n-k}$, where $N^k$ does not contain a line. Later, Cheeger and Gromoll [CG] generalized this to manifolds of nonnegative Ricci curvature, known as the Cheeger-Gromoll splitting theorem. As a consequence, such a manifold has at most two ends (see §2 for the definition of an end). In [A], Abresch studied manifolds with asymptotically nonnegative sectional curvature. He showed that the number of ends of such a manifold is finite and can be estimated from above explicitly. In this note, we consider manifolds with Ricci curvature being nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number. That is, we prove the following theorem.

Theorem. Let $(M^n, o)$ be a Riemannian manifold with base point $o$. If the Ricci curvature is nonnegative outside the geodesic ball $B(o, a)$ of radius $a$ and is bounded from below on $B(o, a)$ by $-(n-1)\Lambda^2$ (for $\Lambda \geq 0$), then there exists a universal bound on the number of ends, e.g.

$$\text{the number of ends of } M^n \leq \frac{2n}{n-1}(\Lambda a)^{-n} \exp \left( \frac{17(n-1)}{2} \Lambda a \right).$$
We learned that P. Li and L. F. Tam proved a similar theorem as an application of the theory of harmonic functions on a complete manifold. Our approach here is more geometrical. A previous version of the Theorem, under the additional condition of a lower bound on the sectional curvature, was proved by Z. Liu. After reading a preliminary version of our paper, Z. Liu informed us that he could also modify his proof, using ideas from this paper, to prove the same theorem as above (see [LT, L]).

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2. IDEA OF THE PROOF OF THE THEOREM

In what follows, we always let $M^n$ be a manifold as in the Theorem.

There are various (but equivalent) definitions of an end of a manifold (cf. [A]), for the sake of our argument, we use the following definition.

**Definition 2.1.** Two rays $\gamma_1$ and $\gamma_2$ starting at the base point $o$ are called cofinal if for any $r > 0$ and any $t \geq r$, $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same component of $M - B(o, r)$. An equivalence class of cofinal rays is called an end of $M$. We will use $[\gamma]$ to denote the class of the ray $\gamma$.

The following proposition is a key to the proof of the theorem.

**Proposition 2.2.** Let $M^n$ be as in the theorem, $[\gamma_1]$ and $[\gamma_2]$ be two different ends of $M^n$, then $d(\gamma_1(4a), \gamma_2(4a)) > 2a$.

Proposition 2.2 will be proved in §3. Assuming it, we now give a proof of the theorem.

**Proof of the theorem.** Let $k$ be an integer and $\gamma_1, \ldots, \gamma_k$ be rays from the base point $o$ going to $k$ different ends. We need to bound $k$ from above. Consider the sphere $S(o, 4a)$ of radius $4a$. Let $\{p_j\}$ be a maximal set of points on $S(o, 4a)$ such that the balls $B(p_j, \frac{1}{2}a)$ are disjoint. Clearly, the balls $B(p_j, a)$ cover $S(o, 4a)$, and since the set $\{\gamma_i(4a), i = 1, \ldots, k\}$ is contained in $S(o, 4a)$, each $\gamma_i(4a)$ is contained in some $B(p_j, a)$. But each ball $B(p_j, a)$ contains at most one $\gamma_i(4a)$ by the Proposition 2.2,
and hence the number of balls is not less than $k$. Thus it suffices to bound the number of balls $B(p_j, \frac{1}{2}a)$.

Notice that

$$B(p_j, \frac{1}{2}a) \subset B(o, \frac{9}{2}a) \subset B(p_j, \frac{17}{2}a).$$

It follows from the Bishop-Gromov volume comparison theorem that

$$\text{vol} B(p_j, \frac{17}{2}a) \leq \int_0^{\frac{17a}{2}} \sinh^{n-1} \Lambda t \, dt \cdot \text{vol} B(p_j, \frac{1}{2}a).$$

Therefore, the number of balls $B(p_j, \frac{1}{2}a)$ is no more than

$$\frac{\int_{\frac{17a}{2}}^{\frac{17a}{2}} \sinh^{n-1} \Lambda t \, dt}{\int_{\frac{1}{2}a}^{\frac{1}{2}a} \sinh^{n-1} \Lambda t \, dt}.$$

Since

$$\frac{\int_0^{\frac{17a}{2}} \sinh^{n-1} \Lambda t \, dt}{\int_0^{\frac{1}{2}a} \sinh^{n-1} \Lambda t \, dt} \leq \frac{2n}{n-1} \frac{17(n-1)}{(\Lambda a)^n},$$

the theorem follows.

Remark 2.3. The bound for the number of ends given here is far from being sharp. An improved bound can be obtained from a more general volume comparison theorem which we can state as follows (for definitions involved, one is refered to [AG]):

A volume comparison theorem. Let $M^n$ be an asymptotically non-negatively Ricci curved manifold. Then for any $p \in M^n$ and for every $0 \leq r \leq R$,

$$\frac{\text{vol} B(p, R)}{\text{vol} B(p, r)} \leq w_n \left( \frac{R}{r} \right)^n$$

where $w_n = (1 + 2u(0)d(o, p))^{n-1} \exp(6(n-1)C_1)$.

Moreover, if $0 \leq r \leq R \leq d(o, p)$ or $2d(o, p) \leq r \leq R$, $w_n$ can be chosen as $2^{2n} \exp(6(n-1)C_1)$ (see [AG] for the definitions of $u(0)$ and $C_1$).

The proof of this theorem will appear elsewhere.

Proof of Proposition 2.2. Let $M$ be a manifold as in the theorem.
For each ray $\gamma$, there is an associated function called the Busemann function, which is defined as follows:

$$b_\gamma(x) = \lim_{t \to \infty} (t - d(x, \gamma(t))).$$

For any given point $p$, let $\alpha_t$ be a minimizing geodesic from $p$ to $\gamma(t)$. As $t \to \infty$, $\alpha_t$ has a convergent subsequence which converges to a ray at $p$. Such a ray is called an asymptotic ray to $\gamma$ at $p$.

Let $\gamma$ be a line. We define $\gamma^+ : [0, \infty) \to M$ by $\gamma^+(t) = \gamma(t)$ and $\gamma^- : [0, \infty) \to M$ by $\gamma^-(t) = \gamma(-t)$.

Let $b^+_\gamma$ ($b^-_\gamma$, resp.) be the associated Busemann function of $\gamma^+$ ($\gamma^-$, resp.).

In [EH], J. Eschenburg and E. Heintze showed, under the assumption that the Ricci curvature is nonnegative everywhere, that $b^+_\gamma$ and $b^-_\gamma$ are smooth harmonic functions with $\text{Hess} b^+_\gamma = 0$ and $b^+_\gamma + b^-_\gamma = 0$. Applying their arguments locally, we can show the following lemma.

**Lemma 3.1.** Let $N$ be the $\delta$-tubular neighborhood of $\gamma$. Suppose that from every point $p$ in $N$, there is an asymptotic ray to $\gamma^+$ and an asymptotic ray to $\gamma^-$ such that the Ricci curvature is nonnegative on both asymptotic rays. Then through every point in $N$, there is a line $\alpha$ which, when parametrized properly, satisfies

$$b^+_\gamma(\alpha^+(t)) = t \quad \text{and} \quad b^-_\gamma(\alpha^-(t)) = t.$$

**Proof.** Let $p$ be any point in $N$. Applying arguments as in the proof of Lemma 3 in [EH], we find that at $p$, $b^+_\gamma + b^-_\gamma = 0$, and $b^\pm_\gamma$ are $C^1$ smooth with $\| \text{grad} b^\pm_\gamma \| = 1$. Hence the asymptotes to $\gamma^\pm$ are uniquely determined at $p$ and fit together to a line, say, $\gamma_p$. Arguments as in the proof of Lemma 2 together with the concluding remarks in [EH] imply that $b^+_\gamma$ ($b^-_\gamma$, resp.) is actually $C^\infty$ smooth with $\text{Hess} b^\pm_\gamma = 0$ on $\gamma_p$. Thus the restriction of $b^\pm_\gamma$ to $\gamma_p$ must be a linear function with derivative 1. After a reparametrization of $\gamma_p$, Lemma 3.1 then follows.

**Remark 3.2.** The same argument as in [EH] of course also implies a local splitting for the metric in $N$, under the assumptions of Lemma 3.1.

**Lemma 3.3.** $M^n$ cannot admit a line $\gamma$ with the following property:

1. $d(\gamma(t), B(o, a)) \geq |t| + 2a \quad \text{for all } t.$
Proof. Suppose there were such a line \( \gamma \). Consider the \( a \)-tubular neighborhood of \( \gamma \). We claim that from any point \( p \) in this neighborhood, all its asymptotic rays to \( \gamma^+ \) (or \( \gamma^- \)) are away from \( B(o, a) \), in particular, the Ricci curvature is nonnegative on such a ray. In fact, let \( s \) be such that \( d(p, \gamma(s)) < a \), then,

\[
d(p, \gamma^\pm(t)) \leq d(p, \gamma(s)) + d(\gamma(s), \gamma^\pm(t)) \\
= d(p, \gamma(s)) + d(\gamma(s), \gamma(\pm t)) \\
\leq a + |s| + t
\]

but any curve from \( p \) to \( \gamma^\pm(t) \) passing through \( B(o, a) \) has length

\[
l \geq d(p, B(o, a)) + d(\gamma^\pm(t), B(o, a)) \\
\geq d(\gamma(s), B(o, a)) + d(\gamma(\pm t), B(o, a)) - a \\
\geq |s| + t + 3a
\]

the last inequality follows from the property (I). Clearly, this implies that any minimizing geodesic, say, \( \alpha_t \), from \( p \) to \( \gamma^\pm(t) \) does not pass through \( B(o, a) \). Hence any convergent subsequence of \( \alpha_t \) will converge to a ray which is away from \( B(o, a) \). This proves the claim.

Next, we claim that through every point of the \( a \)-tubular neighborhood of \( \gamma \), there exists a line with the property (I). Indeed, it follows from the above claim and Lemma 3.1 that through every point of the \( a \)-tubular neighborhood of \( \gamma \), there is a line \( \beta \) such that

\[
b^+_\gamma(\beta^+(t)) = t \quad \text{and} \quad b^-_\gamma(\beta^-(t)) = t.
\]

We need to show that \( \beta \) also has the property (I), i.e.

\[
d(\beta(t), B(o, a)) \geq |t| + 2a \quad \text{for all} \ t.
\]

By symmetry, we may assume that \( t \geq 0 \). Then for any \( r \geq 0 \),

\[
d(\beta(t), B(o, a)) \geq d(\gamma(r), B(o, a)) - d(\beta(t), \gamma(r)) \\
\geq r - d(\beta(t), \gamma(r)) + 2a
\]

(here we used the property (I) for \( \gamma \)). Letting \( r \to \infty \) in the above inequality, we have

\[
d(\beta(t), B(o, a)) \geq b^+_\gamma(\beta(t)) + 2a = t + 2a.
\]

Now let \( \alpha(t) : [0, d] \to M \) be a minimizing geodesic from \( \gamma(0) \) to \( o \), then there is a partition of the interval \([0, d] : t_0 = 0 < t_1 < \cdots < t_k = d \) such that \( d(\alpha(t_i), \alpha(t_{i+1})) < a \).
The last claim implies that there is a line through \( \alpha(t_1) \) with the property (I). Continuing this process inductively, we would find a line with the property (I) through \( \alpha(t_k) \), the base point \( o \), which is absurd.

We are now in the position to prove Proposition 2.2.

**Proof of Proposition 2.2.** Suppose the contrary. That is, \( d(y_1(4a), y_2(4a)) \leq 2a \). Since \([y_1]\) and \([y_2]\) are different ends, there exists an \( A > 4a \) such that \( \gamma_1(t) \) and \( \gamma_2(t) \) are in different unbounded components of \( M - B(o, A) \) for all \( t > A \). Let \( C_t \) \( (t > A) \) be a minimizing geodesic joining \( \gamma_1(t) \) and \( \gamma_2(t) \). Then \( C_t \) must pass through \( B(o, A) \). In addition, we claim that the middle point \( m_t \) of \( C_t \) is in the ball \( B(o, 2A) \). As a matter of fact, let \( p \) be a point in \( C_t \cap B(o, A) \) and without loss of generality we may assume that \( d(p, \gamma_1(t)) \leq d(p, \gamma_2(t)) \), then

\[
\begin{align*}
d(o, m_t) & \leq d(o, p) + d(p, m_t) \\
& \leq A + \frac{1}{2} \rho_t - d(p, \gamma_1(t)) \\
& \leq A + \frac{1}{2} \rho_t - (t - A)
\end{align*}
\]

where \( \rho_t = \) the length of \( C_t \). Notice that

\[
\begin{align*}
\rho_t &= d(\gamma_1(t), \gamma_2(t)) \\
& \leq d(\gamma_1(t), \gamma_1(4a)) + d(\gamma_1(4a), \gamma_2(4a)) + d(\gamma_2(4a), \gamma_2(t)) \\
& \leq 2(t - 4a) + 2a = 2t - 6a.
\end{align*}
\]

Hence,

\[
d(o, m_t) \leq A + \frac{1}{2}(2t - 6a) - (t - A) = 2A - 3a.
\]

This shows that \( m_t \) is in the ball \( B(o, 2A) \).

Now we reparametrize \( C_t \) by translating the origin and with abuse of notation we still denote it by \( C_t \) such that

\[
\begin{align*}
C_t(-\frac{1}{2} \rho_t) &= \gamma_1(t), & C_t(0) &= m_t, & C_t(\frac{1}{2} \rho_t) &= \gamma_2(t).
\end{align*}
\]

We claim that \( C_t(s) \) satisfies property (I) for \(-\frac{1}{2} \rho_t \leq s \leq \frac{1}{2} \rho_t \).

In fact, for any \( s \) (we may assume \( s \geq 0 \)),

\[
d(C_t(s), B(o, a)) \geq d(C_t(\frac{1}{2} \rho_t), B(o, a)) - (\frac{1}{2} \rho_t - s) \geq (t - a) - (t - 3a) + s = s + 2a
\]

where we used the fact \( \rho_t \leq 2t - 6a \). Since \( C_t(0) \in B(o, 2A) \) for all \( t \geq A \), when \( t \to \infty \), a subsequence of \( C_t \) converges to a line \( \gamma(s) \) with the property (I) for all \( s \). (Notice that \( \rho_t \to \infty \), as \( t \to \infty \)). This is a contradiction by Lemma 3.3.
REFERENCES


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