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ENDS OF RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE OUTSIDE A COMPACT SET

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ABSTRACT. We consider complete manifolds with Ricci curvature nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number.

1. INTRODUCTION

Toponogov [T] showed that in a complete manifold of nonnegative sectional curvature, a line splits off isometrically, i.e. any nonnegatively curved M^n is isometric to a Riemannian product $N^k \times R^{n-k}$, where N^k does not contain a line. Later, Cheeger and Gromoll [CG] generalized this to manifolds of nonnegative Ricci curvature, known as the Cheeger-Gromoll splitting theorem. As a consequence, such a manifold has at most two ends (see §2 for the definition of an end). In [A], Abresch studied manifolds with asymptotically nonnegative sectional curvature. He showed that the number of ends of such a manifold is finite and can be estimated from above explicitely. In this note, we consider manifolds with Ricci curvature being nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number. That is, we prove the following theorem.

Theorem. *Let (M^n, o) be a Riemannian manifold with base point o. If the Ricci curvature is nonnegative outside the geodesic ball $B(o, a)$ of radius a and is bounded from below on $B(o, a)$ by $-(n-1)\Lambda^2$ (for $\Lambda \geq 0$), then there exists a universal bound on the number of ends, e.g.*

$$\text{the number of ends of } M^n \leq \frac{2n}{n-1}(\Lambda a)^{-n} \exp\left(\frac{17(n-1)}{2}\Lambda a\right).$$

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We learned that P. Li and L. F. Tam proved a similar theorem as an application of the theory of harmonic functions on a complete manifold. Our approach here is more geometrical. A previous version of the Theorem, under the additional condition of a lower bound on the sectional curvature, was proved by Z. Liu. After reading a preliminary version of our paper, Z. Liu informed us that he could also modify his proof, using ideas from this paper, to prove the same theorem as above (see [LT, L]).

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2. Idea of the proof of the theorem

In what follows, we always let M^n be a manifold as in the Theorem.

There are various (but equivalent) definitions of an end of a manifold (cf. [A]), for the sake of our argument, we use the following definition.

Definition 2.1. Two rays γ_1 and γ_2 starting at the base point o are called cofinal if for any $r > 0$ and any $t \geq r$, $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same component of $M - B(o, r)$. An equivalence class of cofinal rays is called an end of M. We will use $[\gamma]$ to denote the class of the ray γ.

The following proposition is a key to the proof of the theorem.

Proposition 2.2. *Let M^n be as in the theorem, $[\gamma_1]$ and $[\gamma_2]$ be two different ends of M^n, then $d(\gamma_1(4a), \gamma_2(4a)) > 2a$.*

Proposition 2.2 will be proved in §3. Assuming it, we now give a proof of the theorem.

Proof of the theorem. Let k be an integer and $\gamma_1, \dots, \gamma_k$ be rays from the base point o going to k different ends. We need to bound k from above. Consider the sphere $S(o, 4a)$ of radius $4a$. Let $\{p_j\}$ be a maximal set of points on $S(o, 4a)$ such that the balls $B(p_j, \frac{1}{2}a)$ are disjoint. Clearly, the balls $B(p_j, a)$ cover $S(o, 4a)$, and since the set $\{\gamma_i(4a), i = 1, \dots, k\}$ is contained in $S(o, 4a)$, each $\gamma_i(4a)$ is contained in some $B(p_j, a)$. But each ball $B(p_j, a)$ contains at most one $\gamma_i(4a)$ by the Proposition 2.2, and hence the number of balls is not less than k. Thus it suffices to bound the number of balls $B(p_j, \frac{1}{2}a)$.

Notice that

$$B(p_j, \tfrac{1}{2}a) \subset B(o, \tfrac{9}{2}a) \subset B(p_j, \tfrac{17}{2}a).$$

It follows from the Bishop-Gromov volume comparison theorem that

$$\operatorname{vol} B(p_j, \tfrac{17}{2}a) \leq \frac{\int_0^{17a/2} \sinh^{n-1} \Lambda t \, dt}{\int_0^{1a/2} \sinh^{n-1} \Lambda t \, dt} \operatorname{vol} B(p_j, \frac{1}{2}a).$$

Therefore, the number of balls $B(p_j, \frac{1}{2}a)$ is no more than

$$\frac{\displaystyle\int_0^{\frac{17}{2}a} \sinh^{n-1} \Lambda t \, dt}{\displaystyle\int_0^{\frac{1}{2}a} \sinh^{n-1} \Lambda t \, dt}.$$

Since

$$\frac{\displaystyle\int_0^{17a/2} \sinh^{n-1} \Lambda t \, dt}{\displaystyle\int_0^{1a/2} \sinh^{n-1} \Lambda t \, dt} \leq \frac{2n}{n-1} \frac{e^{\frac{17(n-1)}{2}\Lambda a}}{(\Lambda a)^n},$$

the theorem follows.

Remark 2.3. The bound for the number of ends given here is far from being sharp. An improved bound can be obtained from a more general volume comparison theorem which we can state as follows (for definitions involved, one is refered to [AG]):

A volume comparison theorem. *Let M^n be an asymptotically non-negatively Ricci curved manifold. Then for any $p \in M^n$ and for every $0 \leq r \leq R$,*

$$\frac{\operatorname{vol} B(p, R)}{\operatorname{vol} B(p, r)} \leq w_n \left(\frac{R}{r}\right)^n$$

where $w_n = (1 + 2u(0)d(o, p))^{n-1} \; 2^{2n} \exp(6(n-1)C_1)$.

Moreover, if $0 \leq r \leq R \leq d(o, p)$ or $2d(o, p) \leq r \leq R$, w_n can be chosen as $2^{2n} \; \exp(6(n-1)C_1)$ (see [AG] *for the definitions of $u(0)$ and C_1).*

The proof of this theorem will appear elsewhere.

Proof of Proposition 2.2. Let M be a manifold as in the theorem.

For each ray γ, there is an associated function called the Busemann function, which is defined as follows:

$$b_\gamma(x) = \lim_{t\to\infty}(t - d(x, \gamma(t))).$$

For any given point p, let α_t be a minimizing geodesic from p to $\gamma(t)$. As $t \to \infty$, α_t has a convergent subsequence which converges to a ray at p. Such a ray is called an asymptotic ray to γ at p.

Let γ be a line. We define $\gamma^+ : [0, \infty] \to M$ by $\gamma^+(t) = \gamma(t)$ and $\gamma^- : [0, \infty] \to M$ by $\gamma^-(t) = \gamma(-t)$.

Let b_γ^+ (b_γ^-, resp.) be the associated Busemann function of γ^+ (γ^-, resp).

In [EH], J. Eschenburg and E. Heintze showed, under the assumption that the Ricci curvature is nonnegative everywhere, that $b_\gamma^\pm$ are smooth harmonic functions with $\operatorname{Hess} b_\gamma^\pm = 0$ and $b_\gamma^+ + b_\gamma^- = 0$. Applying their arguments locally, we can show the following lemma.

Lemma 3.1. *Let N be the δ-tubular neighborhood of γ. Suppose that from every point p in N, there is an asymptotic ray to γ^+ and an asymptotic ray to γ^- such that the Ricci curvature is nonnegative on both asymptotic rays. Then through every point in N, there is a line α which, when parametrized properly, satisfies*

$$b_\gamma^+(\alpha^+(t)) = t \quad \text{and} \quad b_\gamma^-(\alpha^-(t)) = t.$$

Proof. Let p be any point in N. Applying arguments as in the proof of Lemma 3 in [EH], we find that at p, $b_\gamma^+ + b_\gamma^- = 0$, and $b_\gamma^\pm$ are C^1 smooth with $\|\operatorname{grad} b_\gamma^\pm\| = 1$. Hence the asymptotes to $\gamma^\pm$ are uniquely determined at p and fit together to a line, say, γ_p. Arguments as in the proof of Lemma 2 together with the concluding remarks in [EH] imply that b_γ^+ (b_γ^- ,resp.) is actually C^∞ smooth with $\operatorname{Hess} b_\gamma^\pm = 0$ on γ_p. Thus the restriction of $b_\gamma^\pm$ to γ_p must be a linear function with derivative 1. After a reparametrization of γ_p, Lemma 3.1 then follows.

Remark 3.2. The same argument as in [EH] of course also implies a local splitting for the metric in N, under the assumptions of Lemma 3.1.

Lemma 3.3. *M^n cannot admit a line γ with the following property:*

(I) $$d(\gamma(t), B(o, a)) \geq |t| + 2a \quad \textit{for all } t.$$

Proof. Suppose there were such a line γ. Consider the a-tubular neighborhood of γ. We claim that from any point p in this neighborhood, all its asymptotic rays to γ^+ (or γ^-) are away from $B(o, a)$, in particular, the Ricci curvature is nonnegative on such a ray. In fact, let s be such that $d(p, \gamma(s)) < a$, then,

$$\begin{aligned} d(p, \gamma^{\pm}(t)) &\leq d(p, \gamma(s)) + d(\gamma(s), \gamma^{\pm}(t)) \\ &= d(p, \gamma(s)) + d(\gamma(s), \gamma(\pm t)) \\ &\leq a + |s| + t \end{aligned}$$

but any curve from p to $\gamma^{\pm}(t)$ passing through $B(o, a)$ has length

$$\begin{aligned} l &\geq d(p, B(o, a)) + d(\gamma^{\pm}(t), B(o, a)) \\ &\geq d(\gamma(s), B(o, a)) + d(\gamma(\pm t), B(o, a)) - a \\ &\geq |s| + t + 3a \end{aligned}$$

the last inequality follows from the property (I). Clearly, this implies that any minimizing geodesic, say, α_t, from p to $\gamma^{\pm}(t)$ does not pass through $B(o, a)$. Hence any convergent subsequence of α_t will converge to a ray which is away from $B(o, a)$. This proves the claim.

Next, we claim that through every point of the a-tubular neighborhood of γ, there exists a line with the property (I). Indeed, it follows from the above claim and Lemma 3.1 that through every point of the a-tubular neighborhood of γ, there is a line β such that

$$b_\gamma^+(\beta^+(t)) = t \quad \text{and} \quad b_\gamma^-(\beta^-(t)) = t.$$

We need to show that β also has the property (I), i.e.

$$d(\beta(t), B(o, a)) \geq |t| + 2a \quad for\ all\ t.$$

By symmetry, we may assume that $t \geq 0$. Then for any $r \geq 0$,

$$\begin{aligned} d(\beta(t), B(o, a)) &\geq d(\gamma(r), B(o, a)) - d(\beta(t), \gamma(r)) \\ &\geq r - d(\beta(t), \gamma(r)) + 2a \end{aligned}$$

(here we used the property (I) for γ). Letting $r \to \infty$ in the above inequality, we have

$$d(\beta(t), B(o, a)) \geq b_\gamma^+(\beta(t)) + 2a = t + 2a.$$

Now let $\alpha(t) : [0, d] \to M$ be a minimizing geodesic from $\gamma(0)$ to o, then there is a partition of the inteval $[0, d]$: $t_0 = 0 < t_1 < \cdots < t_k = d$ such that $d(\alpha(t_i), \alpha(t_{i+1})) < a$.

The last claim implies that there is a line through $\alpha(t_1)$ with the property (I). Continuing this process inductively, we would find a line with the property (I) through $\alpha(t_k)$, the base point o, which is absurd.

We are now in the position to prove Proposition 2.2.

Proof of Proposition 2.2. Suppose the contrary. That is, $d(\gamma_1(4a), \gamma_2(4a)) \leq 2a$. Since $[\gamma_1]$ and $[\gamma_2]$ are different ends, there exists an $A > 4a$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are in different unbounded components of $M - B(o, A)$ for all $t > A$. Let C_t $(t > A)$ be a minimizing geodesic joining $\gamma_1(t)$ and $\gamma_2(t)$. Then C_t must pass through $B(o, A)$. In addition, we claim that the middle point m_t of C_t is in the ball $B(o, 2A)$. As a matter of fact, let p be a point in $C_t \cap B(o, A)$ and without loss of generality we may assume that $d(p, \gamma_1(t)) \leq d(p, \gamma_2(t))$, then

$$\begin{aligned} d(o, m_t) &\leq d(o, p) + d(p, m_t) \\ &\leq A + \tfrac{1}{2}\rho_t - d(p, \gamma_1(t)) \\ &\leq A + \tfrac{1}{2}\rho_t - (t - A) \end{aligned}$$

where $\rho_t =$ the length of C_t. Notice that

$$\begin{aligned} \rho_t &= d(\gamma_1(t), \gamma_2(t)) \\ &\leq d(\gamma_1(t), \gamma_1(4a)) + d(\gamma_1(4a), \gamma_2(4a)) + d(\gamma_2(4a), \gamma_2(t)) \\ &\leq 2(t - 4a) + 2a = 2t - 6a. \end{aligned}$$

Hence,

$$\begin{aligned} d(o, m_t) &\leq A + \tfrac{1}{2}(2t - 6a) - (t - A) \\ &= 2A - 3a. \end{aligned}$$

This shows that m_t is in the ball $B(o, 2A)$.

Now we reparametrize C_t by translating the origin and with abuse of notation we still denote it by C_t such that

$$C_t(-\tfrac{1}{2}\rho_t) = \gamma_1(t), \qquad C_t(0) = m_t, \qquad C_t(\tfrac{1}{2}\rho_t) = \gamma_2(t).$$

We claim that $C_t(s)$ satisfies property (I) for $-\frac{1}{2}\rho_t \leq s \leq \frac{1}{2}\rho_t$. In fact, for any s (we may assume $s \geq 0$),

$$\begin{aligned} d(C_t(s), B(o, a)) &\geq d(C_t(\tfrac{1}{2}\rho_t), B(o, a)) - (\tfrac{1}{2}\rho_t - s) \\ &\geq (t - a) - (t - 3a) + s \\ &= s + 2a \end{aligned}$$

where we used the fact $\rho_t \leq 2t - 6a$. Since $C_t(0) \in B(o, 2A)$ for all $t \geq A$, when $t \to \infty$, a subsequence of C_t converges to a line $\gamma(s)$ with the property (I) for all s. (Notice that $\rho_t \to \infty$, as $t \to \infty$). This is a contradiction by Lemma 3.3.

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