SPECTRAL THEORY OF REINHARDT MEASURES

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Let \( \mu \) be a finite positive Borel measure on \( C^n \) \((n \geq 1)\), with compact support \( K \), let \( P^2(\mu) \) be the norm closure in \( L^2(\mu) \) of the algebra of complex polynomials in \( z_1, \ldots, z_n \), and let \( M_z = (M_{z_1}, \ldots, M_{z_n}) \) be the \( n \)-tuple of multiplication operators by the coordinate functions \( z_1, \ldots, z_n \) acting on \( P^2(\mu) \). \( M_z \) is the universal model for cyclic subnormal \( n \)-tuples of operators acting on a separable Hilbert space. For \( n = 1 \), the spectral and algebraic properties of \( M_z \) have been the focus of extensive study (see [Con] for a survey account of the basic results in this area).

One important instance, the case \( d\mu(re^{i\theta}) = d\rho(r) \times \frac{d\theta}{2\pi} \) (where \( \rho \) is a positive Borel measure on \([0, +\infty)\)), gives rise to the class of subnormal weighted shifts, via Berger's Theorem [Con, III.8.16]. Here, the spectral picture of \( M_z \) admits a very simple description:

(i) \( \sigma(M_z) \), the spectrum of \( M_z \), equals \( D_\mu := \{ \lambda \in C : |\lambda| \leq \sup\{|z| : z \in K\} \} \);
(ii) The Fredholm domain of \( M_z \) is \( C \setminus \partial D_\mu \); and
(iii) \( \text{index}(M_z - \lambda) = -1 \) whenever \( \lambda \in \text{int}(D_\mu) \).

The circular symmetry of weighted shifts, reflected in the above description, appears in several variables in the notion of Reinhardt set; \( F \subseteq C^n \) is Reinhardt if \( F = \tau^{-1}(\tau(F)) \), where \( \tau : C^n \to R^n_+ \) is given by \( z \to (|z_1|, \ldots, |z_n|) \). Correspondingly, a compactly supported positive Borel measure \( \mu \) is Reinhardt if it admits a decomposition \( d\mu(re^{i\theta}) = d\rho(r) \times d\theta/(2\pi)^n \), where \( \rho \) is a positive Borel measure on \( R^n_+ \). For instance, volumetric Lebesgue measure on a complete bounded Reinhardt domain \( \Omega \subseteq C^n \) is a Reinhardt measure, in which case \( P^2(\mu) \) is actually \( A^2(\Omega) \), the Bergman space over \( \Omega \).

Received by the editors April 18, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 47A10, 47A53, 47B37, 32A07; Secondary 47B20, 32E20, 47B35, 47A50.

The research of the first author was partially supported by NSF Grant MCS88-0139 and by a University of Iowa Faculty Scholar Award.

The research of the second author was partially supported by NSF Grant DMS 9002969.
The spectral and \( C^* \)-algebraic properties of \( M_z \) on \( A^2(\Omega) \), for \( \Omega \subseteq \mathbb{C}^n \) Reinhardt or pseudoconvex, have been extensively investigated, as have been those of \( M_z \) acting on the Hardy spaces over the Shilov boundary of bounded symmetric domains (e.g., [BC, BCK, BCZ, BdeM, Cob, CM, CS, DH, MR, P, Ra, SSU, U, V]). In this note we announce a complete description of the spectral picture of \( M_z \) in case \( \mu \) is a Reinhardt measure on \( \mathbb{C}^2 \) whose associated weight sequences have limits at infinity in all directions (a notion to be defined later).

To describe our results, we need some notation. Let

\[
V := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\}.
\]

Without loss of generality, we can, and shall, assume that \( K \subseteq \mathbb{D}^n \) and that \( K \) is not contained in \( V \), since otherwise \( M_z \) is unitarily equivalent to the orthogonal direct sum of \( n \)-tuples of the form \((M_{z_1}, \ldots, z_n), 0, \ldots, 0\). For \( \mu \) Reinhardt, the set of bounded point evaluations for \( \mu \) is b.p.e. \( (\mu) := \{\lambda \in \mathbb{C}^n : p \rightarrow p(\lambda), p \in \mathbb{C}[z]\} \), extends boundedly to \( P^2(\mu) \). The Taylor spectrum of \( M_z \), \( \sigma_T(M_z) \), is a nonempty compact subset of \( \mathbb{C}^n \) defined in terms of the exactness of a cochain complex, called the Koszul complex, built from the exterior algebra on \( n \) generators and the coordinates \( M_{z_i}, i = 1, \ldots, n \). The Taylor spectrum enjoys most of the usual properties of the spectrum of a single Hilbert space operator, and supports an analytic functional calculus. There is also a notion of Fredholmness and of index for commuting \( n \)-tuples of operators. (For basic facts on joint spectral systems, the reader is referred to [Cu].) Finally, for a compact subset \( F \) of \( \mathbb{C}^n \), we let \( \hat{F} \) denote the polynomially convex hull of \( F \). If \( F \) is Reinhardt and \( 0 \in F \), then \( \hat{F} = \tau^{-1}\{\exp[\text{convex hull}(\log(\tau(F) \setminus V))]\} \). In particular, if \( z \in F \) then the polydisk \( \{w \in \mathbb{C}^n : |w_i| \leq |z_i|, i = 1, \ldots, n\} \) is contained in \( \hat{F} \).

**Theorem 1.** Let \( \mu \) be a Reinhardt measure on \( \mathbb{C}^n \). Then

(i) \( \text{int} \hat{F} \subseteq \text{b.p.e.}(\mu) \subseteq \hat{F} \);
(ii) \( \sigma_T(M_{\mu}) = \hat{F} \).

To prove Theorem 1(i), we construct a dense-range operator from \( P^2(\mu) \) to the Hardy space of the \( n \)-torus, \( H^2(T^n) \), and we then use it to pull back the Szegö kernel function from \( H^2(T^n) \) to \( P^2(\mu) \); part (ii) requires a spectral inclusion [Cu, Theorem...
7.5(ii)] together with the containment b.p.e.(\mu) \subseteq \sigma_T(M_z). To
discuss our calculation of the Taylor essential spectrum of \( M_z \),
we require some preparations. To begin with, the existence of
bounded point evaluations in a neighborhood \( \Omega \) of the origin
(Theorem 1(i)) gives rise to a kernel function \( k(w, z) \) such that
\[ f(z) = \langle f, k(\cdot, z) \rangle \] for all \( f \in P^2(\mu), \ z \in \Omega. \] Let \( \lambda \in K \setminus V \) and
let \( \varepsilon := (\min_i |\lambda_i|)/3. \) Use of the Cauchy-Schwarz inequality now
yields the following key estimate: There exists a constant \( C > 0 \)
such that
\[ \int_{D(0, \varepsilon) \cap K} |f|^2 \, d\mu \leq C \int_{K \setminus D(0, \varepsilon)} |f|^2 \, d\mu, \]
for every \( f \in P^2(\mu) \), where \( D(0, \varepsilon) \) is the open polydisk centered
at the origin and of multiradius \( (\varepsilon, \ldots, \varepsilon) \). From this we can
derive the next result.

**Proposition 1.** Let \( \mu \) be a Reinhardt measure on \( C^n \). Then \( M_z \) is
(jointly) bounded below, i.e., there exists \( \delta > 0 \) such that
\[ \|z_1 f\|^2 + \cdots + \|z_n f\|^2 \geq \delta^2 \|f\|^2 \] for all \( f \in P^2(\mu) \).

Since \( M_z \) is bounded below, we can use the groupoid machinery
introduced in [CM] to analyze \( C^*(M_z) \). This is done as follows.
First, observe that \( M_z \) is unitarily equivalent to an \( n \)-tuple of
\( n \)-variable weighted shifts; for, if we let \( e_\alpha := z^\alpha /\|z^\alpha\|_{L^2(\mu)} (\alpha \in \mathcal{Z}_n^+), \) it follows from the Reinhardtness of \( \mu \) that \( \{e_\alpha\}_{\alpha \in \mathcal{Z}_n^+} \) is an
orthonormal basis for \( P^2(\mu) \), and that \( M_{z_i} e_\alpha = w_i(\alpha) e_\alpha, \) where
\[ w_i(\alpha) := \|z^{\alpha+i}\| / \|z^\alpha\|, \alpha \in \mathcal{Z}_n^+, i = 1, \ldots, n, \]
\[ i \varepsilon_i := (0, \ldots, 0, 1, 0, \ldots, 0). \]

Similarly, if \( \beta \in \mathcal{Z}_n^+ \), the powers \( M_\beta := M_{z_1}^{\beta_1} \cdots M_{z_n}^{\beta_n} \) are
associated with weight sequences \( w_\beta(\cdot) \). Extend \( w_\beta \) to all of \( \mathcal{Z}_n^+ \)
via \( w_\beta(\alpha) := 0 (\alpha \notin \mathcal{Z}_n^+), \) and let \( \mathcal{A} \) be the closed translation-invariant subalgebra of \( l^\infty(\mathcal{Z}_n^+) \) generated by \( \{w_\beta\}_{\beta \in \mathcal{Z}_n^+}, \) not including the constants. The maximal ideal space of \( \mathcal{A} \), denoted \( Y \), is a noncompact, locally compact Hausdorff space on which
\( \mathcal{Z}_n^+ \) acts by translation. The map \( \varphi: \mathcal{Z}_n^+ \to Y \) given by \( \varphi(\alpha)(a) := a(\alpha), \alpha \in \mathcal{Z}_n^+, a \in Y, \) is injective and open, and \( X := \varphi(\mathcal{Z}_n^+) \subseteq Y \)
is compact. Thus, \( X \) is a suitable compactification of \( \mathcal{Z}_n^+ \) [CM,
Lemma 2.1 and Lemma 2.3. If we let $\mathcal{G} := Y \times \mathbb{Z}^n|_X := \{(y, \alpha) \in Y \times \mathbb{Z}^n; y \in X \text{ and } y + \alpha \in X\}$, we see that $\mathcal{G}$ is the groupoid obtained by reducing the transformation group $Y \times \mathbb{Z}^n$ to $X$, which therefore becomes the unit space of $\mathcal{G}$. A careful analysis of $X$ leads to a detailed description of the ideal structure of $C^*(M^\omega_z)$, based on the correspondence between open invariant subsets of $X$ and closed ideals in $C^*(M^\omega_z)$. Since $X$ is obtained from $\mathbb{Z}^n_+$ by adding suitable limit points at infinity, we need to impose conditions on $\mu$ that guarantee a tractable identification of $X \setminus \mathbb{Z}^n_+$.

We shall say that a Reinhardt measure $\mu$ has convergent weight sequences if for every $i, j = 1, \ldots, n$ and for every $\alpha \in \mathbb{Z}^n_+$, the sequence $\{w_i(\alpha + ke^j)\}_{k=1}^\infty$ is convergent. The following theorem says that one can always assume that $\mu$ has no mass near the origin.

**Theorem 2.** Let $\mu$ be a Reinhardt measure on $\mathbb{C}^n$, let $K := \text{supp } \mu$, assume that $\mu$ has convergent weight sequences, let $\Omega$ be a neighborhood of $\partial \widehat{K}$, and let $\nu := \mu|_{\Omega}$. Then $C^*(M^\omega_z)$ is $*$-isomorphic to $C^*(M^\nu_z)$. Moreover, $M^\omega_z$ is a compact perturbation of $M^\nu_z$ (when each is regarded as an $n$-tuple of $n$-variable weighted shifts on $l^2(\mathbb{Z}^n_+$)). In particular, $M^\omega_z$ and $M^\nu_z$ have identical spectral pictures.

Our description of the spectral picture of $M^\omega_z$ relies on some special properties of the Koszul complex for $M^\omega_z$ in case $n = 2$. Recall that

$$K(M^\omega_z) : 0 \to P^2(\mu) \xrightarrow{D^0(\mu)} P^2(\mu) \oplus P^2(\mu) \xrightarrow{D^1(\mu)} P^2(\mu) \to 0,$$

where

$$D^0(\mu)f = z_1f \oplus z_2f$$

and

$$D^1(\mu)(f \oplus g) = -z_2f + z_1g(f, g \in P^2(\mu)).$$

It follows from Proposition 1 that $D^0(\mu)$ is bounded below, and a trivial calculation then shows that $K(M^\omega_z)$ is exact at the middle stage, so that, by Theorem 1, $\text{index}(M^\omega_z) = 1$ once we establish that $0$ is in the Fredholm domain of $M^\omega_z$. In the sequel, we assume that $n = 2$.

To analyze $X$, we proceed as in [CM]. $\varphi(\mathbb{Z}^2_+)$ is an open invariant subset of $X$, whose associated ideal in $C^*(M^\omega_z)$ is the ideal of compact operators; on the other hand, we let $\infty_G$ denote the subset of $X$ consisting of all limit points of sequences
\{\varphi(k^{(j)})\}_{j=1}^{\infty}$, where $k_j^{(j)} \to +\infty$ for $i = 1, 2$. Clearly $\infty_G$ is a closed invariant subset of $X$, and $\varphi|_{\infty_G} = \infty_G \times \mathbb{Z}^2$. When both $\varphi(V)$ and $\infty_G$ are removed from $X$, we are left with two disjoint subsets, $\infty_N$ and $\infty_E$, consisting of all points in $X$ obtained by taking limits along vertical and horizontal directions, respectively; e.g., $\infty_N := \{x \in X : x = \lim_j \varphi(k^{(j)}) , \{k^{(j)}_1\} \text{ is bounded and } k^{(j)}_2 \to +\infty\}$. In the spectral and algebraic descriptions of $M_z$, the key role is played by $\infty_G$, on which we now focus our attention. Given a direction $\bar{u} \in \mathbb{R}_+^2$, we let

$$\infty_{\bar{u}} := \{x \in X : \exists \{\varphi(k^{(j)})\}_{j=1}^{\infty} \text{ with } \varphi(k^{(j)}) \xrightarrow{w^*} x, k^{(j)} = p_j\bar{u} + q_j\bar{u}^\perp, \text{ and } q_j/p_j \to 0\}.$$

Clearly $X = \bigcup_{\bar{u} \in \mathbb{R}_+^2} \infty_{\bar{u}}$, although two directions may give rise to the same limit points, and different limit points may correspond to the same direction. Nevertheless, the sets $\infty_{\bar{u}}$ carry important information.

A Reinhardt measure $\mu$ on $\mathbb{C}^n$ is said to have convergent weight sequences in all directions (c.w.s.a.d.) if for every direction $\bar{u} \in \mathbb{R}_+^2$ and every sequence $\{k^{(j)} = p_j\bar{u} + q_j\bar{u}^\perp\} \xrightarrow{w^*} x \in \infty_{\bar{u}}$ with $q_j/p_j \to 0$, the convergence of $\{q_j\}$ to some $q \in \mathbb{R}$ implies the convergence of $\{\varphi(k^{(j)})\}_{j=1}^{\infty}$, $i = 1, 2$. Volumetric Lebesgue measure on a complete pseudoconvex Reinhardt domain and surface measure on the boundary of such a domain are two canonical examples of such Reinhardt measures; additional examples are given by Reinhardt measures $\mu$ such that $\text{supp } \mu|_{\partial K} = K \cap \partial \hat{K}$. Intuitively, a measure $\mu$ has c.w.s.a.d. if it admits “balayage” to the boundary. There are, however, measures which do not have c.w.s.a.d.

Following the notation in [SSU], we let $C$ be the closed convex hull of $\log(\tau(K\setminus V))$. Then $\partial C = \partial^0 C \cup \partial^1 C$ (the boundary of $C$ is the union of its 0- and 1-dimensional faces).

**Proposition 2.** Let $\mu$ be a Reinhardt measure on $\mathbb{C}^2$, and assume that $\mu$ has c.w.s.a.d. Then $\infty_G$ can be identified with $\partial C \setminus (F_v \cup F_h)$, where $F_v$ and $F_h$ are the vertical and horizontal (open) faces of $\partial C$, if they exist.

Each oblique 1-dimensional face of $\partial C$ gives rise to a direction $\bar{u} \in \mathbb{R}_+^2$; if $\mu$ has c.w.s.a.d., the corresponding $\infty_{\bar{u}}$ is topologically equivalent to the two-point compactification of the real line, with
the action of $\mathbb{Z}^2$ given by $t + (\alpha_1, \alpha_2) = t + \alpha_1 u_1 - \alpha_2 u_2$. This puts into evidence the presence of a copy of an irrational rotation $\mathbb{C}^*$-algebra when $u_1 / u_2 \notin \mathbb{Q}$, intrinsic to the proof of (iii) below.

**Theorem 3.** Let $\mu$ be a Reinhardt measure on $\mathbb{C}^2$, and assume that $\mu$ has c.w.s.a.d. Then

(i) $M_{\mu} - \lambda$ is bounded below if and only if

$$\lambda \notin (\exp(\partial^0 C \times T^2))^-, $$

(ii) $M_{\mu} - \lambda$ is invertible if and only if $\lambda \notin \hat{K}$,

(iii) $M_{\mu} - \lambda$ is Fredholm if and only if $\lambda \notin \partial \hat{K}$,

(v) $\text{index}(M_{\mu} - \lambda) = \begin{cases} 1 & \text{if } \lambda \in \text{int} \hat{K}, \\ 0 & \text{if } \lambda \notin \hat{K} . \end{cases}$

Theorem 3 should be compared with [SSU, Theorem 1.3], where $\mu$ is volumetric Lebesgue measure on a complete pseudoconvex Reinhardt domain. Unlike the sheaf-theoretical methods used in [CS], [P], and [SSU] for the Bergman space case (obviously not applicable in the case of a general measure), our proof uses J. Bunce's characterization of the left spectrum [B], results from multiparameter spectral theory, and a covering lemma for the spectrum of $C^*(M_{\mu})$ to reduce the problem to the case when $\text{int} \hat{K}$ is the L-shaped domain $\Omega_{\delta_1, \delta_2} := \{(z_1, z_2) \in \mathbb{C}^2 : (|z_1| < \delta_1, |z_2| < 1) \text{ or } (|z_1| < 1, |z_2| < \delta_2) \} \ (0 < \delta_1, \delta_2 < 1)$. For $\Omega_{\delta_1, \delta_2}$, we calculate $\sigma_{\tau_e}(M_{\mu})$ by explicitly exhibiting a pair of (1-variable) bilateral weighted shifts acting on $l^2(\Gamma)$ ($\Gamma$ a subgroup of $\mathbb{R}$), obtained via a suitably built faithful representation of $C^*(M_{\mu})/\mathcal{H}$ associated with the direction $\vec{u} = (\log \delta_2, -\log \delta_1)$. Our techniques also allow us to handle certain cases of Reinhardt measures which do not have c.w.s.a.d., e.g., the example studied in [S].

Details of this work will be forthcoming.

**References**


