A RESTRICTION THEOREM
FOR THE FOURIER TRANSFORM

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ABSTRACT. In this note we will prove a (L^2, L^p) -restriction theorem for certain submanifolds $\mathcal{O}$ of codimension $l > 1$ in an $n-$ dimensional Euclidean space which arise as orbits under the action of a compact group K. As is well known such a result can in general only hold for $1 \leq p \leq \frac{2(n+l)}{n+3l}$. We will show that for the submanifolds under consideration the inequality

$$\int_{\mathcal{O}} |\hat{f}(x)|^2 \, d\mu(x) \leq C \|f\|_p^2$$

holds for $1 \leq p < \frac{2(n+l)}{n+3l}$. Thus we give an answer to a problem stated by J. L. Clerc in [CL, p. 58].

1. INTRODUCTION

The initial published restriction theorem dates from 1970 in the work of C. Fefferman [Fe] where it is attributed to E. M. Stein. In that paper he proved that a (L^2, L^p)-restriction theorem for the unitsphere in $\mathbf{R}^n$ implies sharp results for the Bochner-Riesz multiplier. Since that time one has been interested in the question, for which submanifolds M of $\mathbf{R}^n$ and for which $p, q \geq 1$ the a priori inequality

$$(1.0) \qquad \int_M |\hat{f}(x)|^q \, d\mu(x) \leq C \|f\|_p^q$$

holds. Here $d\mu$ denotes the induced Lebesgue measure on M. An idea of A. Knapp (see [To]) shows that the above inequality can only hold for $1 \leq p < \frac{2n}{n+l}$, $p' \geq \frac{n+l}{n-l} q$, where $l = \operatorname{codim} M$ and $\frac{1}{p'} + \frac{1}{p} = 1$. The restriction of the p-range comes from the fact that the Fourier transform of $d\mu$ can be in $L^{p'}$ only for $p' > \frac{2n}{n-l}$. In case of $M = S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$ and $q = 2$,

Stein and Tomas have shown an optimal result, that is (1.0) holds for $1 \leq p \leq \frac{2(n+1)}{n+3}$. There is also an optimal result for compact hypersurfaces of nonvanishing Gaussian curvature (see [Gr]). The paper of A. Greenleaf [Gr] contains also a result for submanifolds of higher codimension based on the uniform decay in every direction of the Fourier transform of the induced measure with a prescribed order. In general this assumption does not lead to optimal results since radial estimates are inadequate for submanifolds of higher codimension. In many cases such as curves in $\mathbf{R}^n$ or submanifolds whose dimension divides that of the surrounding Euclidean spaces (see [Ch, Pr]) some good results are known, however except in the case of curves it is not known whether or not these are sharp for $q = 2$. A reason for this is that no good estimates for the Fourier transform of the corresponding induced Lebesgue measures are known. If one is interested in the above inequality only for $q = 2$ one has the advantage that a simple computation reduces the matter to the $L^p - L^{p'}$ boundedness for the convolution operator $Tf = \widehat{d\mu} * f$. This can be seen as follows:

$$\int_M |\hat{f}|^2 \, d\mu = \int_M (f * \tilde{f})^\smallfrown \, d\mu \qquad \text{where } \tilde{f} = \bar{f}(-\cdot)$$

$$= \int_{\mathbf{R}^n} \tilde{f} \ \widehat{d\mu} * f \, dx$$

$$\leq \|f\|_p \ \|\widehat{d\mu} * f\|_{p'}.$$

The proof of the results in [Gr, To] (see also [St]) are based on a more or less explicit embedding of the induced Lebesgue measure $d\mu$ in an analytic family of distributions. In our situation in order to obtain optimal results we will construct a suitable analytic family of distributions which contains $\widehat{d\mu_{\mathscr{O}}}$. For $\widehat{d\mu_{\mathscr{O}}}$ we will use a sharp asymptotic estimate which J. L. Clerc has given in [Cl] (see also [DKV]). Up to the endpoint $p = \frac{2(n+l)}{n+3l}$ we answer the question stated by Clerc [Cl, p. 58] which consists basically in the problem whether or not sharp asymptotic estimates for the Fourier transform of induced measures are sufficient to obtain optimal result concerning the restriction problem for the homogeneous submanifolds described below.

2. PRELIMINARIES

Let G be a connected real simple Lie group of noncompact type with finite center, K a maximal compact subgroup, $\mathfrak{g}$ resp.

$\mathfrak{k}$ their Lie algebras and $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ the corresponding Cartan decomposition. The restriction of the adjoint action of G, Ad_G, on K leaves the subspace $\mathfrak{p}$ invariant. Let B be the Killing form on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ gives us an K-invariant scalarproduct on $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, l its dimension and Σ the root system corresponding to the pair $(\mathfrak{g}, \mathfrak{a})$. For $\alpha \in \Sigma$ let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \alpha(H) X, \forall H \in \mathfrak{a}\}$ and $m_\alpha = \dim \mathfrak{g}_\alpha$. Each root $\alpha \in \Sigma$ defines a hyperplane $\alpha(H) = 0$ in the vectorspace $\mathfrak{a}$. These hyperplanes divide $\mathfrak{a}$ into finitely many connected components, the Weyl chambers. We fix such a component and call it $\mathfrak{a}^+$ and set $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha(H) > 0, \forall H \in \mathfrak{a}^+\}$. Let M be the centralizer of $\mathfrak{a}$ in K, M' the normalizer of $\mathfrak{a}$ in K. Then $W = M'/M$ is finite group, called Weyl group. A point in $X \in \mathfrak{p}$ is called regular if its centralizer in $\mathfrak{p}$ is abelian, the set of regular points is a dense open subset $\mathfrak{p}'$ of $\mathfrak{p}$ and the map

$$\beta : K/M \times \mathfrak{a}^+ \ni (kM, H) \to \mathrm{Ad}_k H \in \mathfrak{p}'$$

defines a diffeomorphism (generalized polar coordinates). Furthermore, if we extend β in the second variable in a natural way to the closure of $\mathfrak{a}^+$ we obtain a map onto $\mathfrak{p}$. Let dX, dH denote suitable normalized Euclidean measures on $\mathfrak{p}$ resp. $\mathfrak{a}$ and dk a Haar measure on K.

We define the Fourier transform for $f \in L^1(\mathfrak{p})$ by

$$\hat{f}(X) = \int_\mathfrak{p} f(Y)\, e^{-iB(X, Y)}\, dY.$$

It is an easy fact that for a K-invariant function $f \in L^1(\mathfrak{p})$ their Fourier transform is again K-invariant and we have

$$\hat{f}(X = \mathrm{Ad}_k H) = \int_\mathfrak{p} f(Y)\, J(X, Y)\, dY$$

$$= \int_{\mathfrak{a}^+} f(L)\, J(H, L)\, \omega(L)\, dL,$$

where for $X, Y \in \mathfrak{p}$,

$$J(X, Y) = \int_K e^{-iB(X, \mathrm{Ad}_k Y)}\, dk$$

and ω is a homogeneous function of degree $n - l$ defined by

$$\omega(L) = \prod_{\alpha \in \Sigma^+} \alpha(L)^{m_\alpha}.$$

The function $J(X, Y)$ is called a generalized Bessel function (see [Cl], and for the details of the above [H1, H2]). In [Cl] it was shown that the following sharp asymptotic estimate holds

Proposition 1.1. *Let* $\Lambda \in \mathfrak{a}$ *be a regular point. Then the inequality*

$$|J_\Lambda(H)| \leq C_\Lambda \prod_{\alpha \in \Sigma^+} \frac{1}{(1 + |\alpha(H)|)^{\frac{m_\alpha}{2}}}$$

holds uniformly for $H \in \mathfrak{a}^+$. *Furthermore, the constants* C_Λ *are uniformly bounded if* Λ *lies in a compact subset of* $\mathfrak{a}^+$.

As a consequence of this estimate and the inequality (see [Cl])

$$(1.2) \qquad\qquad |\{H \in \mathfrak{a} \,|\, |\omega(H)| < 1\}| < \infty$$

one obtains

Corollary 1.3. *If* $\Lambda \in \mathfrak{a}^+$ *is regular and* $p > \frac{2n}{n-l}$, $n = \dim \mathfrak{p}$, $l = \dim \mathfrak{a}$. *Then* $J_\Lambda \in L^p(\mathfrak{p})$.

The Corollary implies that the Fourier transform of an K- invariant function in $L^p(\mathfrak{p})$, $1 \leq p < \frac{2n}{n+l}$ is a continuous function on the set of regular points. If we apply this to the group $G = SO(n, 1)$ we get a well-known result for radial functions on $\mathbf{R}^n$. Let us remark that for fixed $\Lambda \in \mathfrak{a}$ $J(\Lambda, X)$ can be considered as the Fourier transform of the measure $d\mu_{\mathscr{O}_\Lambda}$ on the K-orbit of Λ in $\mathfrak{p}$, $\mathscr{O}_\Lambda$, induced by the Lebesgue measure on the Euclidean space $\mathfrak{p}$.

3. THE MAIN RESULT

We will show the following

Theorem 1.4. *Let* $\Lambda \in \mathfrak{a}^+$ *be a regular point and* $1 \leq p < p_0 = \frac{2(n+l)}{n+3l}$. *Then for* $f \in L^1(\mathfrak{p}) \cap L^p(\mathfrak{p})$ *the inequality*

$$\int_{\mathscr{O}_\Lambda} |\hat{f}|^2 \, d\mu_{\mathscr{O}_\Lambda} \leq C \, \|f\|_p^2.$$

holds.

Remark. As mentioned above, up to the endpoint p_0 our result is sharp. If $\mathfrak{g}$ has a complex structure the Bessel functions are known explicitly and we will see how the endpoint can be managed.

For the proof of (1.4) we have to show that $Tf = J_\Lambda * f$, where $J_\Lambda(X) = J(\Lambda, X)$, is a bounded operator from L^p to $L^{p'}$ for $1 \leq p < p_0$. First we fix a C^∞-function ϕ on $\mathfrak{a}^+$ with compact support contained in a small neighborhood of the point Λ which takes the value 1 at Λ and we extend ϕ to a K-invariant C^∞-function on $\mathfrak{p}$ denoted again by ϕ. Since ϕ is identically 1 on

the support of $d\mu_{\mathcal{O}_\Lambda}$ we have $\hat{\phi} * J_\Lambda = J_\Lambda$. Now we define the following analytic family of distributions

$$T_z f = (z + \frac{l}{n-l}) \; \hat{\phi} * (1 + |\omega|)^z \, J_\Lambda * f, \qquad z \in \mathbf{C}.$$

Note that $|\omega(H)|$ is a W-invariant function on $\mathfrak{a}$ and therefore it has a natural extension to $\mathfrak{p}$ as a K-invariant function (see [He 2]), hence T_z is a well-defined convolution operator with a K-invariant kernel. We want to show that T_z is bounded from L^1 to L^∞ for $\operatorname{Re} z = \frac{1}{2}$ and on L^2 for $\operatorname{Re} z < -\frac{l}{n-l}$. By complex interpolation we get our theorem. Concerning the $L^1 - L^\infty$ boundedness we only remark that by (1.1) the kernel is in L^∞ for $\operatorname{Re} z \le \frac{1}{2}$ and convolution with the test function $\hat{\phi}$ leaves L^∞ invariant. For the L^2-boundedness we have to show that the Fourier transform of the kernel of T_z, k_z, is in L^∞ for $\operatorname{Re} z < -\frac{l}{n-l}$. Now, we have for $H \in \mathfrak{a}$,

(∗)

$$|\hat{k}_z(H)| = C \, |\phi(H)| \; | (\, (1 + |\omega|)^z \, J_\Lambda \,)\check{}(H)\,|$$

$$= C \, |\phi(H)| \; \left| \int_{\mathfrak{a}^+} (1 + |\omega(L)|)^z \, J_\Lambda(L) \, J_H(L) \; \omega(L) \, dL \right|.$$

Since H as well as Λ are regular, recall that the support of ϕ lies in a small neighborhood of Λ, we can use the asymptotic estimate (1.1) for J_H and J_Λ. Hence (∗) can be estimated by

$$\le C \int_{\mathfrak{a}} (1 + |\omega(L)|)^{\operatorname{Re} z} \, dL.$$

Using (1.2) a simple homogeneity argument shows that the last integral is finite for $\operatorname{Re} z < -\frac{l}{n-l}$ and our theorem follows.

Let us now see how the endpoint p_0 can be reached if $\mathfrak{g}$ has a complex structure. Under this assumption the generalized Bessel functions are given by (see [Ha, H2]),

$$J_\Lambda(H) = \frac{\text{const.}}{\Pi(H)\,\Pi(\Lambda)} \sum_{w \in W} \det w \; e^{iB(\Lambda, wH)}$$

for $H, \Lambda \in \mathfrak{a}$, where $\Pi(H) = \sqrt{\omega(H)} = \prod_{\alpha \in \Sigma^+} \alpha(H)$. Now we plug this nice formula into (∗), change to polar coordinates in $\mathfrak{a}$, use Fubini's theorem and the fact that the Fourier transform of a smooth function which behaves at infinity essentially like a homogeneous function of degree $-1 + ir$ is essentially a homogeneous function of degree $-ir$, and we see that $|\hat{k}_z(H)|$ is bounded for

$\mathrm{Re}\, z = -\frac{l}{n-l}$ by a constant which increases only polynomially for $\mathrm{Im}\, z \to \infty$. As before a complex interpolation gives what we want.

4. Final remark

It is shown in [Mo] that in general a (L^2, L^{p_0}) restriction theorem for compact submanifolds M of codimension l imply that a compactly supported function $m_\alpha \in C^\infty(\mathbf{R}^n \setminus M)$ which has a singularity of the form $\mathrm{dist}(x, M)^\alpha$ near M, defines a multiplier for L^p if $\alpha > n(\frac{1}{p} - \frac{1}{2}) - \frac{l}{2}$ and $1 \le p \le p_0$. Moreover for the homogeneous submanifolds discussed above this result is sharp (up to cases where m_α is a smooth function on $\mathfrak{p}$).

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