A CLASSIFICATION OF COHERENT STATE REPRESENTATIONS OF UNIMODULAR LIE GROUPS

WOJCIECH LISIECKI

1. INTRODUCTION

Let $G$ be a connected Lie group and $(\pi, \mathcal{H})$ a unitary representation of $G$ on a complex Hilbert space $\mathcal{H}$. Throughout we shall assume that $(\pi, \mathcal{H})$ is nontrivial in the sense that $\dim \mathcal{H} > 1$. By a coherent state orbit (CS orbit for short) for $(\pi, \mathcal{H})$ we mean a complex orbit of $G$ on the projective space $\mathbb{P}(\mathcal{H})$ (which is equipped with a natural structure of an (infinite-dimensional in general) Kaehler manifold (cf. [L])). We call $(\pi, \mathcal{H})$ a coherent state representation (CS representation for short) if (1) it admits a CS orbit, (2) is irreducible and (3) has (at most) discrete kernel, and we call $G$ a CS group if it possesses CS representations. The purpose of this note is to announce a complete classification of connected unimodular CS groups and their CS representations (Theorems 1 and 2 below). This generalizes the results of Enright-Howe-Wallach [EHW] and Jakobsen [J] on the classification of unitary highest weight (or holomorphic) representations of reductive groups (which coincide with the CS representations as we have shown in [L]). The proofs are "geometric," the main tool being the recent structure theory of homogeneous Kaehler manifolds due to Dorfmeister and Nakajima [DN].

In physics, any orbit on $\mathbb{P}(\mathcal{H})$ is called a system of coherent states in the sense of Perelomov (see [P] and the references therein).

Of particular importance are symplectic coherent state orbits; in many cases such an orbit may be interpreted as the classical phase space of the system whose quantum phase space is $\mathbb{P}(\mathcal{H})$. Such an embedding of the classical phase space into the quantum one is the starting point of Berezin's quantization (see [B1] and [B2];
see also [T] for a comparison of Berezin's quantization with the Kostant-Souriau geometric quantization and the "quantization of states" proposed recently by Odzijewicz (see [O1] and [O2]). In both theories, the case of complex orbits plays a distinguished role. On one hand, "complex" coherent states are in a sense closest to the classical states [P] and on the other, we may apply in this case powerful techniques of complex analysis (with Bergman type reproducing kernels playing an essential role).

Thus there is a strong physical motivation for studying CS representations.

2. BASIC PROPERTIES OF CS REPRESENTATIONS

Here the term CS representation refers to a \((\pi, \mathcal{H})\) which has property (1) but not necessarily (2) and (3).

**Proposition 1** [L]. Any CS orbit has a natural structure of a Hamiltonian G-space and the corresponding moment mapping takes it diffeomorphically onto an integral coadjoint orbit with Kaehler (i.e. positive totally complex) polarization.

There is a natural holomorphic line bundle \(E\) over \(P(\mathcal{H})\) whose fiber at \([v] = Cv\) is the dual \([v]^*\). The linear dual \(\mathcal{H}^*\) of \(\mathcal{H}\) is naturally isomorphic to the space of holomorphic sections of \(E\). Given a CS orbit \(G \cdot [v]\) corresponding to a CS representation \((\pi, \mathcal{H})\), we get a natural map from \(\mathcal{H}^*\) to the space \(\Gamma(G \cdot [v], L)\) of holomorphic sections of \(L\), the restriction of \(E\) to \(G \cdot [v]\).

**Proposition 2.** The following are equivalent.

(i) \(v\) is a cyclic vector for \((\pi, \mathcal{H})\).
(ii) The map \(\mathcal{H}^* \to \Gamma(G \cdot [v], L)\) is injective.
(iii) \((\pi, \mathcal{H})\) is irreducible.

The implications (i) \(\Rightarrow\) (ii) and (iii) \(\Rightarrow\) (i) are clear, and (ii) \(\Rightarrow\) (iii) can be deduced from a well-known theorem of Kobayashi [K].

3. THREE SPECIAL CASES

It turns out that the case of a general unimodular group can be reduced to three special cases, which we shall now briefly discuss.

3.1. Heisenberg groups. Let \(H_n\) be a \((2n + 1)\)-dimensional Heisenberg group (not necessarily simply connected). Identify the (multiplicative) group \(X(C)\) of unitary characters of the center \(C\) of \(H_n\) with an (additive) subgroup of the dual \(c^*\) of the Lie
algebra of $C$. The infinite-dimensional irreducible unitary representations of $H_n$ are in 1-1 correspondence with the nonzero elements $\lambda$ of $X(C)$, $(\beta_\lambda, \mathcal{F}_\lambda)$ being the unique (up to equivalence) representation with $\lambda$ as central character (or, in other terms, the unique representation corresponding, via Kirillov's bijection, to the integral coadjoint orbit $\mathcal{O}_\lambda$ determined by $\lambda$). It is well known that any $(\beta_\lambda, \mathcal{F}_\lambda)$ is a CS representation. Any of the CS orbits on $\text{P}(\mathcal{F}_\lambda)$ is mapped by its moment onto $\mathcal{O}_{-\lambda}$. This establishes a 1-1 correspondence between these orbits and Kaehler polarizations of $\mathcal{O}_\lambda$ which, in turn, are in 1-1 correspondence with points of the Siegel space $\mathfrak{S}_n$ (i.e. the Hermitian symmetric space $\text{Sp}(2n, \mathbb{R})/\text{U}(n)$).

Next we consider reductive groups. We shall say that a reductive group is of compact (resp. noncompact) type if its Lie algebra is so.

### 3.2. Groups of compact type [KS]

Any such group is a CS group and any of its nontrivial representations is a CS representation. For any CS representation, there is exactly one CS orbit, namely the orbit through a highest weight line. Geometrically, these orbits are compact simply connected homogeneous Kaehler manifolds (i.e. flag manifolds).

### 3.3. Groups of noncompact type [L]

Such a group is a CS group if and only if it is of Hermitian type (i.e. the symmetric space $\mathfrak{D}$ associated with it is of Hermitian type). CS representations are the highest weight representations. Again the orbit through a highest weight line is the unique CS orbit for a given CS representation. Geometrically, it is a holomorphic fiber bundle over $\mathfrak{D}$ (equipped with one of its invariant complex structures) with flag manifolds as fibers.

### 4. Homogeneous Kaehler manifolds

Our approach to the problem of classifying CS groups is based on Dorfmeister-Nakajima theorem [DN] (which gives an affirmative answer to a long standing conjecture of Vinberg and Gindikin). For our purposes, it is convenient to state it as follows. Every homogeneous Kaehler manifold $X$ has a holomorphic double fibration

$$
\begin{align*}
X & \\
\downarrow & \\
M & \to \mathfrak{D},
\end{align*}
$$

where $M$ is a homogeneous Kaehler manifold without flat homogeneous Kaehler submanifolds and the fibers of $X \to M$ are flat.
homogeneous Kaehler manifolds (i.e. they are of the form $\mathbb{C}^n/\Gamma$, where $\Gamma$ is a discrete subgroup of $\mathbb{C}^n$ and the Kaehler metric is induced by the standard Kaehler metric on $\mathbb{C}^n$), $\mathcal{D}$ is a homogeneous bounded domain and the fibers of $M \to \mathcal{D}$ are flag manifolds. Such a double fibration is unique and is preserved by all automorphisms of $X$.

5. STRUCTURE OF A CS ORBIT

Now suppose $(\pi, \mathcal{H})$ is a CS representation of $G$ and $X = G \cdot [v] \subset \mathbb{P}(\mathcal{H})$ is a CS orbit such that neither its flat fibers nor $\mathcal{D}$ reduce to points. The fact that $X$ is a Hamiltonian $G$-space implies that these flat fibers are isomorphic to some $\mathbb{C}^n$ and coincide with the orbits of a Heisenberg group $N$ (of dimension $2n + 1$) which is contained in $G$ as a normal subgroup. Let $J_N$ denote the moment mapping of the corresponding Hamiltonian action of $N$ on $X$. Since the orbits of this action are symplectic, the symplectic reduction theorem (see [AM]) implies that $J_N(X)$ is a single coadjoint orbit $\mathcal{O}_\lambda$.

$N$ being a normal subgroup of $G$, there is a homomorphism

$$\tilde{\rho}: G \to \text{Aut}(N), \quad g \mapsto \text{Int} g|_N$$

(where $\text{Int} g$ denotes the inner automorphism of $G$ corresponding to $g$), which factors through $N$ to give a homomorphism

$$\rho: S = G/N \to \text{Out} N = \text{Aut} N/\text{Int} N.$$

It is clear that $\tilde{\rho}(S) \subset (\text{Aut } N)_\lambda$, the stabilizer of $\mathcal{O}_\lambda$ (or $\lambda$) in $\text{Aut } N$ (which is the same for all $\lambda \neq 0$), and, consequently,

$$\rho(S) \subset (\text{Aut } N)_\lambda/\text{Int } N \cong \text{Sp}(2n, \mathbb{R}).$$

Being a complex submanifold of $X$, each $N$-orbit carries a Kaehler polarization which is mapped by $J_N$ into a Kaehler polarization of $\mathcal{O}_\lambda$. We thus get a $\rho$-equivariant map from the orbit space $M = X/N$ to the space of all Kaehler polarizations of $\mathcal{O}_\lambda$, i.e. the Siegel space $\mathcal{G}_n$. It can be shown that this map is holomorphic. Hence it factors through the compact fibers of $M$ to give a $\rho$-equivariant holomorphic map

$$\rho_\mathcal{D}: \mathcal{D} \to \mathcal{G}_n.$$
show that then $S = G/N$ is also unimodular and so is its quotient $S/N$ which acts effectively on $\mathcal{G}$. Moreover, $N$ is of compact type (here the assumption that $\pi$ has discrete kernel is essential).

Now a theorem of Hano [Ha] asserts that if a unimodular Lie group acts effectively and transitively on a bounded domain, then it is semisimple and the domain is symmetric. Thus $S/N$ is semisimple and, consequently, $S$ is reductive and of Hermitian type. It follows that $N$ coincides with the nilradical (the maximal connected nilpotent normal Lie subgroup) of $G$.

We have sketched the proof of the "only if part" of the following.

**Theorem 1.** A connected unimodular (nonreductive) Lie group $G$ is a CS group if and only if it satisfies the following conditions.

(i) The nilradical $N$ of $G$ is isomorphic to a Heisenberg group $H_n$.

(ii) The reductive group $S = G/N$ is either of compact or of Hermitian type and its image under the natural homomorphism $\rho: S \to \text{Out}(N)$ is contained in $\text{Sp}(2n, \mathbb{R})$.

(iii) If $S$ is of Hermitian type, there exists a $\rho$-equivariant holomorphic map from the Hermitian symmetric space $\mathcal{D}$ associated with $S$ to the Siegel space $\mathfrak{G}_n$.

That this theorem really classifies unimodular CS groups follows from the results of Satake (see [S2]) who classified $\rho$-equivariant holomorphic maps $\mathcal{D} \to \mathfrak{G}_n$ (this classification is closely related to the classification of Howe's reductive dual pairs in $\text{Sp}(2n, \mathbb{R})$ (cf. [Ho]).

7. Classification of CS Representations

Irreducible unitary representations of the groups which occur in Theorem 1 have been classified by Satake [S1]. Using his results and the results of the preceding sections (with Proposition 2 playing an essential role) we can complete the proof of Theorem 1 and also prove the following.

**Theorem 2.** Suppose $G$ has properties (i)–(iii) of Theorem 1. For any nonzero $\lambda \in X(C)$, let $(\sigma, \mathcal{F})$ be a projective representation of $G$ obtained by composing the (projective) metaplectic representation of $(\text{Aut} N) \setminus \{\beta, \mathcal{F}\}$ with $\tilde{\rho}$ and let $\alpha$ be its cocycle ($\alpha$ does not depend on $\lambda$ and can be considered as a cocycle on $S = G/N$). Let $(\pi, \mathcal{E})$ be an irreducible projective
unitary representation of $S$ with the following properties:

(i) its cocycle is $\alpha^{-1}$;
(ii) its kernel $\ker \pi_1$ is contained in $N_S$ (cf. §6);
(iii) the corresponding representation of $S/\ker \pi_1$ is a (projective) CS representation.

Then $(\pi, H)$, where $H = \mathcal{H} \otimes \mathcal{F}$ (Hilbert tensor product) and

$$\pi(g) = \tilde{\pi}_1(g) \otimes \sigma_\lambda(g) \quad \text{for } g \in G,$$

$\tilde{\pi}_1$ being the composition of $\pi_1$ and the projection $G \to S$, is a (linear) CS representation of $G$ and any CS representation of $G$ is of this form.

REFERENCES


Institute of Mathematics, University of Warsaw, Białystok Branch, Akademicka 2, 15-267 Białystok, Poland