

G-ELLIPTIC SYSTEMS AND THE GENUS ZERO PROBLEM FOR M_{24}

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1. INTRODUCTION

Toward the end of the 1970s J. H. Conway and S. P. Norton wrote a quite unique and influential paper [3] in which they produced incontrovertible evidence, based partly on ideas of J. McKay and J. G. Thompson [13], that the Fischer-Griess Monster simple group M was intimately related to the modular function $j(\tau)$ familiar from the theory of modular forms. Thus was so-called 'Moonshine' brought into the world. Among other things, it was conjectured that there is a natural infinite-dimensional complex vector space V^{\sharp} which carries a \mathbf{Z} -grading into finite-dimensional subspaces $V^{\sharp} = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots$ such that each V_i is a CM -module. Moreover the graded character $\text{ch}_{V^{\sharp}}(q) := \sum_{n \geq -1} \dim V_n q^n$ of V^{\sharp} should be the Fourier expansion of $j(\tau)$, namely $q^{-1} + 196884q + 21,493,760q^2 + \cdots$ if the indeterminate q is interpreted as $e^{2\pi i\tau}$ for τ in the upper half-plane H . There should in addition be an analogous interpretation for

$$\text{tr}_{V^{\sharp}}(g, q) := \sum_{n \geq -1} \text{trace}_{V_n}(g) q^n$$

for each $g \in M$; namely, corresponding to g there is a discrete subgroup Γ_g of $SL_2(\mathbf{R})$ of genus zero i.e., the compactified orbit space $(\Gamma_g \backslash H)^*$ is a sphere, such that $\text{tr}_{V^{\sharp}}(g, q)$ is a modular function invariant under Γ_g which generates the field of all such functions over \mathbf{C} . (Briefly, $\text{tr}_{V^{\sharp}}(g, q)$ is a so-called 'hauptmodul'.)

This conjecture is known as the genus zero problem for M and has recently been settled in the affirmative. The work of Frenkel-

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Lepowsky-Meurman [7] and Borchers [1] established the existence of a V^\sharp with the correct graded character and, equally importantly, showed that the proper setting for studying such questions is conformal field theory and the theory of chiral algebras. Very recently, Borchers [2] has announced a solution of the remainder of the genus zero problem for M by indeed verifying the hauptmodul property of $\text{tr}_{V^\sharp}(g, q)$, but it is perhaps fair to say that the true meaning of this property is still something of a mystery.

In the meantime, Norton has considerably extended the genus zero problem. In an appendix to [12], he suggested the possibility of attaching to each element $g \in M$ a certain complex vector space V_g^\sharp which is graded (by rational numbers with bounded denominator) into finite-dimensional $C_M(g)$ -modules ($C_M(g) = \{h \in M \mid gh = \{g\}\}$), so that V_1^\sharp is our original space V^\sharp and for $h \in C_M(g)$ the q -expansion $\text{tr}_{V_g^\sharp}(h, q)$ is either a constant or again a hauptmodul. Furthermore there is a certain ‘compatibility’ between the various trace functions afforded by the modular group $SL_2(\mathbf{Z})$ —see below for more details.

This conjecture remains open, but it has been pointed out by physicists (cf. [5, 14] for example) that there is again a ‘stringy’ perspective from which to understand it: conformal field theory on an orbifold. Very roughly, if a finite group G acts smoothly on a manifold X , one considers the space of all smooth (in a suitable sense) maps $S^1 \rightarrow G \backslash X$. This decomposes into subsets indexed by $g \in G$ corresponding to maps $f: [0, 1] \rightarrow X$ such that $f(1) = gf(0)$. After implementation of a suitable quantization procedure, what results is a Hilbert space H_g which enjoys all of the properties of V_g^\sharp sketched above except perhaps the hauptmodul property of the trace functions, which appears to be a relatively unusual occurrence.

Thus this point of view leads to the tantalizing question of whether the Monster M acts on a suitable (24-dimensional?) manifold X in such a way that the extended genus zero problem of Norton can be explained. Furthermore, it suggests developing a moonshine type theory for *arbitrary* finite groups. Do other of the so-called sporadic groups (cf. [8]) enjoy monstrous properties? The results of this announcement are a step in this direction. We show how, given any finite group $G \leq SO(24n, \mathbf{R})$, there is attached a certain family $\{H_g\}$ of explicitly described complex vector spaces, with all of the expected properties. Furthermore,

in at least one case, $M_{24} \leq SO(24, \mathbf{R})$ where M_{24} is the largest sporadic Mathieu group [loc cit.], the extended genus zero property holds. One of the novelties of our approach, however, is that the construction of the family $\{H_g\}$ proceeds not from a group action on a manifold but from a quite different perspective. We adopt the vantage point of classical number theory and following [11] introduce a theory of Hecke-type operators into the study of G -elliptic systems (see below), so that for us the modular invariance of the trace functions—a crucial issue in conformal field theory—depends on the theory of these operators.

2. STATEMENT OF RESULTS

For a finite group G , let $P(G) = P = \{(g, h) \in G \times G | gh = hg\}$. If $\Gamma = SL_2(\mathbf{Z})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, define

$$(2.1) \quad (g, h) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (g^a h^c, g^b h^d).$$

Then the map $\alpha: (g, h) \mapsto (g, h) \cdot \alpha^{-1}$ defines a left action of Γ on P . Let \mathcal{M} be the space of nowhere-vanishing \mathbf{C} -valued functions on P , which we can naturally regard as a (multiplicative) right $\mathbf{Z}\Gamma$ -module.

Let $k: P \rightarrow \mathbf{Z}$ be a Γ -invariant weight function and $j(\alpha, \tau) = (c\tau + d)$ the usual 1-cocycle of Γ associated to its left action on H , where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then for a 1-cocycle σ in $C^1(\Gamma, \mathcal{M})$ there is a right action of Γ on functions $f: P \times H \rightarrow \mathbf{C}$ given by

$$(2.2) \quad (f|_{k, \sigma} \alpha)(g, h; \tau) = \sigma(\alpha; g, h) j(\alpha, \tau)^{-k(g, h)} f((g, h)^{-1} \alpha; \alpha\tau)$$

where in (2.2) we have set $\sigma(\alpha; g, h) = \sigma(\alpha)((g, h))$.

Definition. A G -elliptic system consists of the following data:

- (i) A weight function k and 1-cocycle σ as above.
- (ii) For each $g \in G$, a (rationally) graded virtual complex vector space

$$H_g = \bigoplus_{n \in \mathbf{Z}} H_g^n q^{n/D_g}$$

with H_g^n a virtual $C_G(g)$ -module, q indeterminate, D_g an integer depending only on g . Moreover the following

are required to be satisfied:

If $(g, h) \in P$ and $f(g, h; \tau) := \sum \text{tr}_{H_g^n}(h)q^{n/D_g}$ is the graded trace of h on H_g , $q = e^{2\pi i\tau}$, then

(iii) $f(g, h; \tau)$ is meromorphic on $H \cup \{\infty\}$.

(iv) $f|_{k, \sigma}\alpha = f$ for all $\alpha \in \Gamma$.

(v) $f(g, h; \tau) = f(g^x, h^x, \tau)$ for all $x \in G$.

It is not hard to see that (iii) and (iv) already force $f(g, h; \tau)$ to be a modular form of weight $k(g, h)$, and a G -elliptic system is essentially a collection of modular forms (and their q -expansions at the cusps) together with an interpretation as trace functions. A discussion of this can be found in [9]. Orbifolds provide examples where each H_g is a *genuine* representation space, the weight function k is identically zero, and each $\sigma(\alpha; g, h)$ is a root of unity. Norton's extended genus zero problem for the Monster is that such an elliptic system exists for M —together with the hauptmodul property.

We say that the G -elliptic system $\{H_g\}$ is of *weight zero* if the weight function k is identically zero i.e., if each trace function $\text{trace}(h \text{ on } H_g)$ is a modular function of weight zero. The G -elliptic systems we are interested in describing are of this type and depend only on an embedding $G \leq SO(24d, \mathbf{R})$, d an integer.

Let V be the underlying orthogonal space with scalars extended to \mathbf{C} , and set $W = V \oplus V$.

Fix $g \in G$. Using the shorthand $e(x)$ for $\exp(2\pi ix)$, let $0 < b_1 < b_2 < \dots < b_r \leq 1$ be such that $e(b_j)$ are the *distinct* eigenvalues of g in its action on V , and let V_j (resp. W_j) be the corresponding eigenspace. We define graded $C_G(g)$ -modules as follows:

$$(2.3) \quad M_g = \bigoplus_{j=1}^r \bigoplus_{i=0}^{\infty} W_j q^{i+b_j-1/2}$$

$$(2.4) \quad N_g = \bigoplus_{j=1}^r \bigoplus_{i=0}^{\infty} W_j q^{i+b_j}$$

and a linear (degree 1) character given by

$$(2.5) \quad \lambda_g(h) = \det \cdot h^{-1} \text{ on } \bigoplus_{b_j > 1/2} V_j, \quad h \in C_G(g).$$

In (2.3), $*$ indicates that the sum runs only over indices satisfying $i + b_j > 1/2$.

As W_1 and $W_{1/2}$ have a structure of real orthogonal spaces admitting the diagonal action of $C_G(g)$, we may consider the usual $1/2$ -spin modules Δ_g^\pm for the corresponding real spin group affording representations of $C_G(g)$. Similarly Δ_g is the restriction of the spin-module for the real spin group corresponding to W_1 .

Next, let $B_2(X) = X^2 - X + 1/6$ denote the second Bernoulli polynomial, and set

$$(2.6) \quad B(g) = \frac{1}{4} \sum_{j=1}^r \dim V_j B_2(b_j).$$

For a (graded) module M let $\Lambda^\pm(M)$ denote the even and odd parts of the exterior algebra on M , that is $\Lambda^+(M) = \bigoplus_{k \geq 0} \Lambda^{2k}(M)$, $\Lambda^-(M) = \bigoplus_{k \geq 0} \Lambda^{2k+1}(M)$.

Finally, if $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the standard generators of Γ and $h \in C_G(g)$, define

$$(2.7) \quad \sigma(T; g, h) = e \left\{ \sum_{\substack{j \\ b_j < 1/2}} b_j^2 \dim V_j - \frac{1}{8} \dim V_{1/2} \right\}$$

$$(2.8) \quad \sigma(S; g, h) = e \left\{ -2 \sum_{\substack{j \\ b_j < 1/2}} \sum_{i=1}^{\dim V_j} b_j b'_{j_i} - \frac{1}{4} \dim V_{1/2, 1/2} \right\}$$

where in (2.8), $e(b'_{j_1}), e(b'_{j_2}), \dots$ runs over the eigenvalues of h (with multiplicity) on V_j and $-1/2 < b_{j_i} \leq 1/2$, and $V_{1/2, 1/2}$ is the subspace of V on which both g and h act as -1 .

Remark. We have $\sigma(S; g, g)^{-1} = \sigma(T; g, h)^2$. For the relevance of this to orbifold theory, see [4].

Theorem 1. *The function σ defined by (2.7) and (2.8) extends to a 1-cocycle $\sigma \in C^1(\Gamma, \mathcal{M})$.*

Theorem 2. *For $g \in G$, set*

$$(2.9) \quad H_g = q^{B(g^2) - 2B(g)} (\Delta_g^+ \otimes \Lambda^+(M_g) \oplus \Delta_g^- \otimes \Lambda^-(M_g)) \\ + 1/2q^{2B(g)} \lambda_g \otimes \Delta_g \otimes \Lambda(N_g).$$

Then $\{H_g\}$ is an elliptic system of weight zero corresponding to the 1-cocycle σ of Γ given by Proposition 1.

Remarks. (1) There is a ‘denominator’ of 2 in (2.9). This can be removed, yielding a true representation space for $C_G(g)$ if, for example Δ_g is the sum of a pair of isomorphic $C_G(g)$ -modules. This in turn holds, for example, if G fixes a nonzero vector in V ; in particular, this is the case in Theorem 3.

(2) The character of H_1 i.e. $f(1, 1, \tau)$ is a monic polynomial in $j(\tau)$ of degree d (recall $\dim V = 24d$). More precisely, we have

$$(2.10) \quad f(1, 1, \tau) = \text{char } H_1 = T_2 \Delta^d / \Delta^d$$

where $\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ is the discriminant and T_2 is the second Hecke operator. There is an analogous, but more complicated, identity for the other trace functions.

Theorem 3. *Take $G = M_{24} \leq SO(24, \mathbf{R})$ and let $\{H_g\}$ be as in Theorem 2. Then this elliptic system has the genus zero property. Precisely, if $(g, h) \in P(M_{24})$ and $f(g, h; \tau) = \text{trace}(h \text{ on } H_g)$ then one of the following holds:*

- (a) $\langle g, h \rangle$ has noncyclic Sylow 2-subgroup and $f(g, h; \tau) = 0$.
- (b) $\langle g, h \rangle$ has cyclic Sylow 2-subgroup, and if $\Gamma_{g, h}$ is the full invariance group of $f(g, h; \tau)$ is $SL_2(\mathbf{R})$ then $(\Gamma_{g, h} \backslash H)^*$ is a sphere and the function field is generated by $f(g, h; \tau)$.

3. SKETCH OF PROOFS

Keeping earlier notation, set

$$(3.1) \quad L_g = q^{B(g)} \Lambda \left(- \bigoplus_{j=1}^r \bigoplus_{n=0}^{\infty} V_j q^{n+b_j} \right).$$

Lemma 1. *Assume that G acts on V evenly i.e., for each $(g, h) \in P$ we have $\dim C_V(\langle g, h \rangle)$ even. Then $\{L_g\}$ is a (virtual) elliptic system with respect to a certain $\beta \in C^1(\Gamma, \mathcal{M})$ and weight function $k: \langle g, h \rangle \mapsto 1/2 \dim C_V(\langle g, h \rangle)$. (A discussion of a special case is already contained in [10].)*

We wish to define a theory of Hecke-type operators which will act on $\{L_g\}$ and produce a second elliptic system for us. This

entails extending (2.2) to actions of $\alpha \in GL_2(\mathbf{Z})$. There is a left action of $GL_2(\mathbf{Z})$ extending (2.1), namely

$$(3.2) \quad \alpha: (g, h) \mapsto (g, h)\alpha', \alpha' = (\det \alpha)\alpha^{-1}$$

but to extend (2.2) we must introduce an index in the fashion of Jacobi forms (cf. [6]). Precisely, the 1-cocycle β of Lemma 1 factors as $\gamma\delta$ where $\gamma \in C^1(\Gamma, \mathcal{M})$ and δ is a *twisted 1-cocycle* in the sense that

$$(3.3) \quad \delta(\alpha_1\alpha_2; g, h) = \delta(\alpha_1; (g, h)\alpha_2')\delta(\alpha_2; g, h)^{\det \alpha_1}$$

for $\alpha_1, \alpha_2 \in GL_2(\mathbf{Z})$. Note that $\delta(\alpha; g, h)$ may well vanish, but not if $\det \alpha = 1$. Now define for $f: P \times H \rightarrow \mathbf{C}$,

$$(3.4) \quad f\|_{k, m, \gamma, \delta}\alpha(g, h; \tau) = (\det \alpha)^{1/2k(g, h)} j(\alpha, \tau)^{-k(g, h)} \\ \times f((g, h)\alpha'; \alpha\tau)\gamma(\alpha; g, h)\delta(\alpha; g, h)^m.$$

We have

$$(3.5) \quad f_{k, m, \gamma, \delta}\|\alpha\beta = (f\|_{k, m, \gamma, \delta}\alpha)\|_{k, m\det \alpha, \gamma, \delta}\beta$$

in case either (i) $\det \alpha\beta$ is odd or (ii) V is the sum of a pair of isomorphic CG-modules (e.g., replace V by W). Under either of these conditions we can develop a theory of Hecke operators. To some extent this is formal: for a double coset $\Gamma\alpha\Gamma$ of Γ in $GL_2(\mathbf{Z})$ define

$$(3.6) \quad f\|_{k, m, \gamma, \delta}[\Gamma\alpha\Gamma](g, h; \tau) \\ = (\det \alpha)^{k(g, h)/2-1} \sum_i f\|_{k, m, \gamma, \delta}\alpha_i(g, h; \tau)$$

where α_i runs over representatives of $\Gamma\backslash\Gamma\alpha\Gamma$, which is well defined by (3.5). According as cases (i) or (ii) above take n odd or arbitrary and set

$$(3.7) \quad T_n f = \sum f\|_{k, m, \gamma, \delta}[\Gamma\alpha\Gamma]$$

with the sum running over all double cosets of Γ in the matrices of determinant n in $GL_2(\mathbf{Z})$. Then by (3.5) we get

$$(3.8) \quad T_n f\|_{k, n, \gamma, \delta}\alpha = T_n f, \quad \alpha \in \Gamma$$

if f is the trace function of the elliptic system $\{L_g\}$ of Lemma 1. The hard part is to now establish

Proposition 2. *There is a G -elliptic system $\{E_g\}$ and a function $\varepsilon: P(G) \rightarrow \{\text{roots of unity}\}$ with the following properties:*

- (a) *The weight function is the same as that for $\{L_g\}$.*
- (b) *The 1-cocycle $\gamma' \in C(\Gamma, \mathcal{M})$ associated with $\{E_g\}$ is given by $\beta' = \gamma_\varepsilon \gamma \delta^n$ where γ_ε is the 1-coboundary determined by ε .*
- (c) *The trace function $F(g, h; \tau)$ of $\{E_g\}$ satisfies $F = \varepsilon^{-1} T_n f$.*

Theorem 2 follows from the case $n = 2$ by taking the quotient elliptic system $\{E_g\}/\{L_g\}$ (note: the denominator never vanishes!) while the 1-cocycle σ of Theorem 1 is just $\gamma_\varepsilon \delta$. Because we have complete control over the various 1-cocycles, i.e. they are computable in terms of the action of G on V , the functional equation (3.8) for $T_2 f$ and the analogue for f itself lead to the proof of Theorem 3.

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