The author stops short of the Hardy-Littlewood method and its powerful application to Waring's problem and other number theoretic representation problems.

Wilf's book is beautifully written. The witty and slightly folksy style (one useful technique is called "the Snake Oil Method," because it has so many applications) hides the real depth of many of the results. There are masses of examples either worked out in the book or left for the reader. In the latter case the solutions are given in the back of the book. Anyone who enjoys combinatorics problems, or who likes messing about with power series and seeing what identities can be obtained that way, will get much pleasure from this book.

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Computability; (*computable functions, logic, and the foundations of mathematics*), by Richard L. Epstein and Walter A. Carnielli. Wadsworth & Brooks/Cole, Pacific Grove, California, 1989, 295 pp. ISBN 0-534-10356-1

Euler knew perfectly well what a function was. It was defined by an expression showing the operations to be performed in order to obtain the value for a given argument. These operations could involve limits, but nevertheless the expressions had a clear computational meaning. Indeed, for Euler and his contemporaries, integrals and series made it possible to "discover" hitherto unknown functions: elliptic integrals, complex exponentials, Bessel functions. But there was a problem with trigonometric series. For example, the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

had what appeared to be a general solution as a trigonometric series, while it was evidently satisfied by

$$Af(x+ct) + Bf(x-ct)$$

for an "arbitrary" function f. Clearly an "arbitrary" function could not be the sum of periodic functions. When Fourier made it plain that arbitrary functions (defined on $(-\pi, \pi]$) could indeed be expanded into trigonometric series, it was plain that the notion of function that had served Euler was no longer adequate for mathematical analysis. It was in this context that Dirichlet moved decisively towards set theory as the foundation for mathematics by proposing what has become our standard notion of function: an arbitrary correspondence.

Given this history, it seems entirely appropriate that modern set theory began with Canto's investigations of uniqueness conditions for trigonometric series. Cantor was led to iterate the process of forming the set of limit points of a given set of real numbers:

$$E, E', E'', \ldots.$$

Moreover, in the case where this sequence does not eventually become constant, Cantor was led to form their intersection, E^{ω}, and then continue the iteration. Having thus pushed into the transfinite, there was no turning back for Cantor, who proceeded to develop the first coherent mathematical theory of the actual infinite. While these developments were embraced by many mathematicians, there were others for whom this departure from the clear computational content of the mathematics of Euler was unacceptable. Kronecker was the first important mathematician to attack Cantor's *Mengenlehre* but, with the realization that Cantor's methods seemed to lead to outright contradictions, there were many others: Poincaré, Weyl, and most important, Brouwer. Brouwer's *intuitionism* proposed a radical return to a mathematics with a constructive computational content, willingly abandoning the transfinite. To Hilbert, this was a call to arms. The proposed expulsion from Cantor's "paradise"[1] was not to be countenanced.

Hilbert accepted Brouwer's requirement that ultimately it was explicit computational verifiability that was needed for the foundations of mathematics. But Hilbert was prepared to be bound by this limitation to *finitistic* methods only in his proposed *proof theory*. This proof theory was to lead to consistency proofs for formal logical systems *within* which the full strength of Cantorian set theory could be formalized, consistency proofs even Brouwer would have to accept. From today's perspective it is difficult to

[1] p. 51 (page references are to "readings" in the book being reviewed).

comprehend the vehemence of the discussion as Hilbert and Brouwer thundered defiance at one another[2]:

Brouwer: ··· nothing of mathematical value will be attained in this manner; a false theory which is not stopped by a contradiction is none the less false, just as a criminal policy unchecked by a reprimanding court is none the less criminal.

Hilbert: ··· Weyl and Brouwer are... trying to establish mathematics by pitching overboard everything that does not suit them and setting up an embargo. ... The effect is to dismember our science and run the risk of losing a large part of our most valuable possessions. ... Today the State is thoroughly armed through the labors of Frege, Dedekind, and Cantor. The efforts of Brouwer and Weyl are foredoomed to futility.

To the logical positivists of the Vienna Circle meeting in the late 1920s, the constructivism apparently accepted by Brouwer and Hilbert was a necessary defense against meaningless metaphysical notions. The young Kurt Gödel attended the meetings of the Vienna Circle, but did not accept their point of view. In attempting to provide consistency proofs of the kind Hilbert was seeking, Gödel was led to distinguish the *truth* of a statement of elementary number theory from its *provability* in a particular formal system, a distinction that would have been regarded as meaningless by most participants in the Vienna Circle[3]. Once this distinction was clearly made, it was not difficult for Gödel to show that the provable statements in any appropriate formal system can never include all true statements of elementary number theory. A further conclusion was that such systems are never strong enough to permit the proof of their own consistency. Since Hilbert's goal was the proof of the consistency of such systems using only very weak finitistic methods, Gödel's results seemed to destroy Hilbert's program, although Gödel himself held out the possibility that methods could be judged finitistic although not formalizable within the systems in questioin might be found[4].

Indeed, Hilbert's proof theory continues to flourish. Consistency proofs can be given which use Hilbert's finitistic methods augmented only by specific combinatorial principles concerning

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[2]Quoted from E. T. Bell, *Development of mathematics*, second ed., McGraw-Hill 1945, pp. 569–570.

[3]p. 216.

[4]pp. 213–214.

whose finitistic character differences of opinion are possible. The best known principle of this kind was introduced by Gentzen in 1936 to prove the consistency of **PA**, a formalization of elementary number theory. The principle in question involves the transfinite ordinal number ε_0 which is the limit of the sequence

$$\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \cdots$$

and is therefore the smallest solution of the equation $\omega^x = x$. One can readily define a (computable) relation $\prec$ which is a well-ordering of the natural numbers of order type ε_0. Gentzen's principle from which he showed how to prove the consistency of **PA** is simply the principle of transfinite induction for $\prec$: for any condition which can be formulated in **PA**, if the condition holding for all $y \prec x$ implies that it must also hold for x, then it must hold for all natural numbers.

Nevertheless, the truth is that Hilbert's program for the foundations of mathematics never really recovered after Gödel's results, and the doctrinaire pronouncements of the 1920s concerning foundational issues seem a bit naïve today. Perhaps the most coherent position is the straightforward Platonism espoused by Gödel[5], which accepted the set theoretic foundation for mathematics and insisted the evidence for the "existence" of such abstract entities was as compelling as that for physical objects. Probably most working mathematicians think in Platonic terms, but would, if pressed, offer some kind of naïve formalism as their underlying philosophy.

Meanwhile, Brouwer's student Heyting developed a formal logical calculus intended to embody the proof methods regarded as intuitionistically acceptable. This meant that intuitionism itself became susceptible to investigation by mathematical methods. A surprising result was that although intuitionistic logic was conceived as being narrower than classical logic, there was a sense in which classical logic was a sub-theory of intuitionistic logic. Namely a logical formula written entirely in terms of $\neg$, $\wedge$, and $\forall$ turns out to be valid classically if and only if it is valid intuitionistically. But, classically all other logical operations are definable in terms of these:

$$p \Rightarrow q = \neg(p \wedge \neg q)\,;\ (\exists x) = \neg(\forall x)\neg\,,$$

although of course these definitions are not intuitionistically acceptable. None of this has prevented intuitionistic logic from be-

[5] p. 11, pp. 226–227.

coming of interest in theoretical computer science. Since an intuitionistic proof of the existence of some object is guaranteed to be constructive, it is in principle possible to systematically "compile" such a proof into a computer program for actually generating the object in question. This suggests the very tempting goal of a software methodology that automatically produces practical computer programs, guaranteed to be correct, from existence proofs.

Our students remain for the most part blissfully ignorant of all of these foundational matters. The book being reviewed attempts to do something about this by presenting a smorgasbord of writings on the foundations of mathematics from Plato to Gödel. These are presented in the context of one of the great triumphs of foundational investigations: computability theory, otherwise known as recursion theory. By what Gödel has called "a kind of miracle",[6] it has turned out to be possible to give a precise mathematical characterization of the class of functions whose values can be calculated by means of an algorithm. Moreover, many apparently different characterizations turn out to yield the very same class of functions (the authors, somewhat hyperbolically, call this "the most amazing fact"). The resulting theory has provided a mathematical theory of digital computers, and has made it possible to show that various mathematical problems (the word problem for groups, Hilbert's tenth problem) are algorithmically unsolvable.

The book being reviewed is a kind of sandwich. A textbook on computability theory and undecidablity in arithmetic is placed between two more philosophical parts. The book begins with a discussion of the paradoxes of self-reference and the mathematical contexts in which they arise, with many readings and exercises, and it closes with a presentation of various attitudes towards constructivity in mathematics, including those of Brouwer and Bishop, in the light of computability theory. Most students will probably find the readings rather tough going, and will need the help of a patient teacher. But it could be lots of fun. (There is apparently an instructor's manual; but I have not seen it.)

There is a technical error regarding Gentzen's consistency proof for first order arithmetic on p. 214. The authors insist that Gentzen's induction principle for $\prec$ "cannot be formalized" within first order arithmetic, saying that "$\cdots$ if it could then by adding that [sic] further axiom(s) to **PA** we would have a theory which

[6] p. 227.

could prove its own consistency." Since $\prec$ is arithmetically definable, Gentzen's induction principle can be expressed as a "schema" in the language of **PA** in the same manner as ordinary induction. In the system obtained by adding this principle to **PA** (call it **PA+**), Gentzen's consistency proof for **PA** can certainly be carried out. But this is not an instance of a "theory which could prove its own consistency;" the consistency of **PA** is proved in a different system **PA+**. There is also a misstatement on p. 215: the authors surely meant to say that it *was* clear to Gödel that the primitive recursive functions were *not* "all the computable ones... ."

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Linear operators in spaces with an indefinite metric, by T. Ya. Azizov and I. S. Iokhvidov. John Wiley & Sons, 1989, 300 pp., $82.95. ISBN 0-471-92129-7

Let H be a Hilbert space (over the complex numbers), and let J be a bounded linear selfadjoint operator on H such that $J^2 = I$. Consider the sesquilinear form $[\cdot, \cdot]$ induced by J:

$$[x, y] = \langle Jx, y\rangle, \qquad x, y \in H,$$

where $\langle\cdot, \cdot\rangle$ stands for the scalar product in H. The corresponding quadratic form $[x, x]$ is indefinite (unless $J = I$ or $J = -I$), in other words, there exist $x, y \in H$ for which $[x, x] < 0$ and $[y, y] > 0$. The space H, together with the sesquilinear form $[\cdot, \cdot]$ generated by some J as above, is commonly called a *Krein space*. One can also define the Krein spaces intrinsically, by starting with a topological vector space and a continuous sesquilinear form on it, and by imposing suitable completeness and nondegeneracy axioms. The reviewed book is devoted to the geometry of Krein spaces and the spectral structure and related properties of several important classes of bounded and unbounded linear operators on Krein spaces.

1. THE SUBJECT

Why Krein spaces? As with many mathematical disciplines, there are two compelling reasons: (1) important applications in