was astonished to learn that the Pell equation $x^2 - Dy^2 = 1$ had been solved as early as 3000 years ago in India. The fundamental solution of $x^2 - 8y^2 - 1$ is $9 - 8 \cdot 1 = 1$. For $D = 13$, the fundamental solution is $421201 - 13 \cdot 32400 = 1$. Since this solution appears in an ancient writing, an algorithm was known in prehistoric times.

In modern times, the idea of finding solutions in positive integers has been extended. For all sorts of polynomial and exponential equations, one looks for solutions in (some fixed) number field. It is believed that the only solution of $3^x - 2^y = 1$ is given by $9 - 8 = 1$, but I know no proof. If $a, b, c > 1$, the equation $a^x - b^y = c^z$ $(x, y, z > 2)$ probably has only finitely many solutions. The famous Fermat equation is a special case. It is conjectured to have no solutions.

The Thue equation is $f(x, y) = k$, $k > 0$. Here $f(x, y)$ is a homogeneous polynomial in x, y with (say) integer coefficients. Shorey and Tijdeman proved that each such equation has only finitely many solutions. De Weger gives an algorithm for finding all solutions. For reasonably small values of k the algorithm is practicable with current technology.

Another problem is: solve $x + y = z^2$, where the prime divisors of xy are restricted to be among $2, 3, 5, 7$. De Weger tabulates all solutions. The largest is $2^3 3^7 5^4 7 + 1 = 13^2 673^2$. Other large ones are $2^{12} + 3^3 \cdot 5 \cdot 7 = 71^2$, $2^5 3^2 5^2 7 + 1 = 449^2$. The smallest are $3 + 1 = 4$, $2 + 2 = 4$.

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Volterra integral and functional equations, by G. Gripenberg, S.-O. Londen, and O. Staffans. Cambridge University Press, Cambridge, 1990, 701 pp., $99.50. ISBN 0-521-37289-5

The spirit of this book is captured in the following quotation from the introduction "The subject of Volterra equations in finite dimensional space has reached a certain maturity, which both motivates and makes possible a coherent presentation." The authors of the book were all active participants in carrying the subject to that maturity, and the depth of their understanding is clearly displayed. It is truly the definitive work on the subject considered.

The book is not easy reading. Many of the theorems are technically complicated even to state, let alone prove. Much of this complication stems from the effort to treat Volterra equations with kernels which are measures. The bonus from this is a systematic study of measure type convolutions, transforms of these and the associated harmonic analysis. There is a wealth of information on this subject. Many of the results are new, at least to this reviewer, and should appeal to analysts who may have only a passing interest in Volterra equations.

It is helpful to approach this book with a little historical background. Volterra equations deal with systems that have a "memory effect," that is ones in which the past history of the state variables must be considered. This is to be contrasted to differential equations in which only present time values count. The memory notion goes back at least to Boltzmann and Volterra and arises very naturally in for instance, population models with age-dependence, viscoelastic materials, and control systems with feedback.

The modern theory, that of this text, really began in the 1960s with several diverse but related studies. There was work by Hale on delay equations and early studies by Levin and Nohel on Volterra equations. Control theory ideas, based on transform techniques and called frequency domain methods entered through the work of Popov, Halanay, and Corduneanu. Another stimulus came from continuum mechanics with the studies by Coleman, Gurtin, and Noll on the "fading memory" concept for materials. A natural outgrowth of the mechanics studies was the need to study Volterra equations on infinite dimensional spaces. This prompted a long line of investigations by Dafermos, Hrusa, Nohel, Slemrod, this reviewer, and many others.

The unifying feature of all the work above is the observation that memory effects can endow systems with an internal damping which is crucial to stability questions.

The authors of the text under review were all involved in the infinite dimensional studies but their major interest has been to develop and extend the results for finite dimensional equations and that is the subject of this book.

The book itself contains extensive references and no effort to list them will be made here. There are, however, four books which should be mentioned. The first is by Hale [1] and considers delay equations. The second by Corduneanu [2] treats frequency domain methods. A third by Miller [3] is a precursor of the text under review. Finally there is a recent book by Hrusa, Renardy, and Nohel [4] which treats the infinite-dimensional, mechanics problems and is an indispensable complement to the reviewed book.

The equations under study in G-L-S are of two types and can be written in the very general form,

$$u(t) = g(t, u^t) \tag{I}$$

$$\dot{u}(t) = g(t, u^t). \tag{II}$$

Here u is a map from R into $\mathbf{R}^n$ and u^t denotes its history, $u^t(\tau) = u(t-\tau)$. For each t, $g(t, \cdot)$ is a map from some space of histories into $\mathbf{R}^n$. Most of the book is concerned with the situation in which g has a special form namely it is the composition of a linear memory operator and a fixed but possibly nonlinear mapping h. More precisely,

$$g(t, u^t) = \int_0^\infty a(t, \tau) h(u(t-\tau))\, d\tau. \tag{1}$$

When $a(t, \tau, h(u^t)) = a(t-\tau, h(u^t))$ one says (1) is of convolution type and the main portion of the book is devoted to this case.

If g in (II) has the form (1) with a Dirac measure (II) becomes a differential equation (no memory). One must then append to (II) an initial condition $u(t_0) = u_0$. In the more general form (1), however, (II) requires an initial history u^{t_0} that is a knowledge of u from $t = -\infty$ up to time t_0.

The first three hundred pages of the book deal with (I) and (II) under the assumption (1) with h linear. Most of this is further specialized to the convolution situation. In this case one can think of the initial history u^0 on $(-\infty, 0]$ as producing a forcing term on $t > 0$ and the equations have the form,

$$u(t) = \int_0^t d\mu(\tau) u(t-\tau) = f(t) \qquad t > 0 \tag{I$'$}$$

$$\dot{u}(t) = \int_0^t d\mu(\tau) u(t-\tau) = f(t) \qquad t > 0. \tag{II$'$}$$

Now one has available Laplace transforms techniques and these are systematically studied in the first eight chapters.

Equations of the type (I'), (II') admit of solution formulas involving resolvents which are solutions for special right-hand sides. The damping idea is equivalent to having these kernels be integrable on $(0, \infty)$ (the Paley-Wiener theorem). This question is of paramount interest in the text. Here μ may be simply a measure and in order even to consider the resolvent question one has to make sense out of the equations, the resolvent equations and the transforms of both. As the authors show, the resolvent question is more difficult for (I') than for (II') since in the former case one must make sense out of convolutions of measures. The first four chapters consider these questions. The reader should pay particular attention to the careful proof of the Paley-Wiener theorem in $\mathbf{R}^n$ and to the appendices on transforms of measures and convolutions.

In Chapter four the authors consider the resolvent question for (I'). They phrase it in the language of weighted spaces. This is a central notion in continuum mechanics fading memory theory. The authors mention this but do not stress it and this reviewer would suggest that this chapter should be read in conjunction with the references by Coleman and Mizel.

Chapters five and six contain some results on special types of kernels. The first results deal with completely monotone kernels which were, historically, the first ones considered. Chapter six treats the Shea-Wainger theory concerning kernels which do not decay fast enough to be in L^1. This chapter contains some very subtle harmonic analysis ideas.

The remaining chapters of the first part concern a variety of questions for linear equations. Chapter seven considers equations which permit asymptotic instability and is very much in the spirit of the work on delay equations in Hale's book. Chapter eight contains a brief discussion of semigroup ideas. To put Volterra equations into this concept one must think of semigroups over the history space, again as for delay equations. Chapter nine contains a discussion of kernels of nonconvolution type. The results are interesting but, in this reviewer's opinion, not really too usable.

The remainder of the book deals with nonlinear problems. This is approached first, in Chapters ten and eleven, through perturbation theory. Here the integrability of the resolvent comes into play. These results are technically rather complicated but there are no real surprises. Things proceed pretty much as for ordinary differential equations. Chapter thirteen is again work that needs to be done but is not surprising. It contains existence, uniqueness and continuous dependence results for nonlinear equations. The ideas are really those of ordinary differential equations and the results are local.

The real meat of Volterra theory is contained in the remainder of the book. Here the questions concern global behavior (without assumptions of small data). These questions are: Does a global solution exist and is it asymptotically stable? These are the key ideas which can be carried over to the infinite dimensional case as can be seen in H-N-R book.

Chapter fourteen begins the global study with Lyapunov techniques which, historically, were the basis for the pioneering work of Levin and Nohel. It was observed by Levin and Nohel that for the case of smooth kernels, $d\mu(\tau) = a(\tau)\,d\tau$ stability in (I′) and (II′) could be produced by imposing conditions on the sign of a and some of its derivatives and monotonicity conditions on g. This idea is extended here to the more situation.

Chapter fifteen provides the general setting for asymptotic theory including the idea of limit sets. A very interesting (and very nontrivial) question for Volterra type equations is finding appropriate limit equations to determine asymptotic limits. This question is thoroughly explored in Chapter fifteen.

Chapters sixteen through eighteen represent the generalization of frequency domain methods, as described in Corduneanu's book. It was long known in circuit analysis that necessary and sufficient conditions for stability of linear circuits involved the positivity of the real part of the Laplace transforms of transition matrices in the open right half transform plane. It turns out that a sharpening of this result to positivity in the closed half-plane is sufficient for asymptotic stability, not only for linear equations but for nonlinear versions of (I′) and (II′) when h is monotone. This is the idea of strongly positive kernels and it is systematically studied here. Once again this requires deep results in harmonic analysis on measures. The outcome is very rewarding. It yields a set of very practical criteria for establishing asymptotic stability particularly for feedback control. It is quite interesting that one is able to show that the frequency domain conditions are implied by the sign conditions of Chapter fourteen but not conversely. Thus the frequency domain conditions represent a real extension.

Chapters nineteen and twenty provide some special results on combinations of Lyapunov and frequency domain methods as well as a discussion of the special case of log convex kernels.

The book does what it says it will do. It provides a coherent presentation. It is a demanding but rewarding book on a subject which has been extensively explored over the past thirty years. The patient reader will come away with a real sense of the accomplishments of these explorations.

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Fractals everywhere, by Michael F. Barnsley. Academic Press, New York, 1988, 394 pp., $39.95. ISBN 0-12-079062-9

Fractal books have always been blockbusters, at least in terms of sales, and the book under review (aimed at undergraduates) seems to be no exception. Filled with color illustrations, lots of figures, examples, exercises, and, above all, a style of writing more closely associated with the advertising industry than with mathematical work, this book is superficially a very attractive buy for a student wanting to know about fractal geometry. In fact the book covers only one branch of fractal geometry known as 'iterated function systems,' but does cover this thoroughly, and in a highly unusual way.

I will try to judge what this book achieves for the student reader and for the professional mathematician, as well as the impression it makes for fractal geometry within mathematics.