arithmetic of number fields will eventually have to learn something about Hilbert modular forms. As an introduction to the analytic and arithmetic aspects of the subject, Garrett's book may be the best place to start.

REFERENCES

[BR] D. Blasius and J. Rogawski, *Galois representations for Hilbert modular forms*, Bull. Amer. Math. Soc. (N.S.) **21** (1989), 65–70.

[G] P. Garrett, *Pullbacks of Eisenstein series: Applications*, Automorphic Forms of Several Complex Variables, Birkhaüser-Verlag, Boston, 1984.

[Ge] S. Gelbart, *Automorphic forms on Adéle groups*, Princeton Univ. Press, Princeton, NJ, 1975.

[GGP] I. M. Gelfand, M. I. Graev, and I. I. Piatetski-Shapiro, *Representation theory and automorphic functions*, W. B. Saunders Co., Philadelphia, 1969.

[J] H. Jacquet, *Automorphic forms on* $GL(2)$. II, Lecture Notes in Math., vol. 278, Springer-Verlag, Berlin and New York, 1972.

[JL] H. Jacquet and R. P. Langlands, *Automorphic forms on* $GL(2)$, Lecture Notes in Math., vol. 114, Springer-Verlag, Berlin and New York, 1970.

[Sa] I. Satake, *Spherical functions and Ramanujan conjecture*, Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, RI, 1966.

[Sh] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten, Japan and Princeton Univ. Press, Princeton, NJ, 1971.

[T] R. Taylor, *On Galois representations associated to Hilbert modular forms*, Invent. Math. (1989).

MICHAEL HARRIS
BRANDEIS UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 1, July 1991

0273-0979/91 $1.00 + $.25 per page

Coxeter graphs and towers of algebras, by F. M. Goodman, P. de la Harpe, and V. F. R. Jones. MSRI Publications #14, Springer-Verlag, Berlin and New York, 1989, 286 pp., $32.00. ISBN 0-387-96979-9

Coxeter graphs and towers of algebras could not appear at a more auspicious moment. One of the authors, Vaughan F. R. Jones, has just been awarded the Fields medal, and (if we may presume to infer the reasons for the choice of Jones by the Fields Medal Committee), the decision had much to do with the subject of this monograph.

In 1984 Jones discovered an astonishing relationship between a construction in Von Neumann Algebras and a theorem in geometric topology [Jo,2]. As a result, he found a new polynomial invariant for knots and links in 3-space. His invariant had been missed completely by topologists, in spite of intense activity in closely related areas during the preceding 60 years, and was a complete surprise. As time went on, it became clear that it had to do in a bewildering variety of ways with widely separated areas of mathematics and physics. These included (in addition to knot theory) Dynkin diagrams and the representation theory of simple Lie algebras, also that part of statistical mechanics having to do with "exactly solvable models," and also the very new area of quantum groups.

The central connecting link in all this mathematics was a tower of nested algebras which Jones had discovered some years earlier in the course of proving (see [Jo,1]) a theorem which is known as the *Index Theorem*. The monograph under review has dual purposes. First, it was written as a research monograph for the experts, reviewing in one place the literature which relates to the Index Theorem, in many places writing down proofs for the first time in full generality and with suitable detail. At the same time, it is an elementary introduction to that part of Von Neumann algebras which concerns the Index Theorem and its relationship to Dynkin diagrams. Thus it is very very timely.

The monograph is almost entirely algebraic in nature, and directed at topics in Von Neumann algebras. On the other hand, it was written after the momentous nature of Jones' discoveries was beginning to be appreciated, and the authors had that in mind when they wrote the book. Thus a largely successful effort has been made to make the material accessible to a broad audience. All the basic definitions are there, and the tools which are used are built up from little more than basic linear algebra and the notion of a norm. As a nonexpert, we could follow details of the proofs whenever we put in the required effort. However, make no mistake—if you want to read this book, you will have to work hard, and to know where you are going.

In order to explain the subject matter, we discuss the Index Theorem, briefly. Let $\mathbf{M}$ denote a Von Neumann algebra. Thus $\mathbf{M}$ is an algebra of bounded operators acting on a Hilbert space $\mathfrak{H}$. The algebra $\mathbf{M}$ is called a *factor* if its center consists only of scalar multiples of the identity. The factor is *type* II_1 if it admits a linear functional, called a trace, tr: $\mathbf{M} \to \mathbf{C}$, which satisfies the following three conditions:

$$\mathrm{tr}(\mathbf{xy}) = \mathrm{tr}(\mathbf{yx}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbf{M}$$
$$\mathrm{tr}(\mathbf{1}) = 1 \quad \text{(normalized)}.$$
$$\mathrm{tr}(\mathbf{xx}^*) > 0 \quad \text{for all } \mathbf{x} \in \mathbf{M}, \quad \text{where } \mathbf{x}^* \text{ is the adjoint of } \mathbf{x}.$$

In this situation it is known that the trace is unique, in the sense that it is the only linear form satisfying the first two conditions. An old discovery of Murray and Von Neumann was that factors of type II_1 provide a type of "scale" by which one can measure the dimension $\dim_{\mathbf{M}}(\mathfrak{H})$ of $\mathfrak{H}$. The notion of dimension which occurs here generalizes the familiar notion of integer-valued dimensions, because for appropriate $\mathbf{M}$ and $\mathfrak{H}$ it can be any nonnegative real number or ∞. (The interested reader will find a good introduction to this interesting topic in the monograph under review.)

The starting point of the work in [Jo,1] is the following question: if $\mathbf{M}_1$ is a type II_1 factor and if $\mathbf{M}_0 \subset \mathbf{M}_1$ is a *subfactor*, is there any restriction on the real numbers which occur as the ratio

$$\lambda = \dim_{\mathbf{M}_0}(\mathfrak{H}) / \dim_{\mathbf{M}_1}(\mathfrak{H})?$$

The question has the flavor of questions one studies in Galois theory. On the face of it, there was no reason to think that λ could not take on any value in $[0, \infty]$, so Jones' answer came as a complete surprise. He called λ the *index* $[\mathbf{M}_1 : \mathbf{M}_0]$ of $\mathbf{M}_0$ in $\mathbf{M}_1$, and proved a type of rigidity theorem about type II_1 factors and their subfactors:

The Jones Index Theorem. *If* $\mathbf{M}_1$ *is a* II_1 *factor and* $\mathbf{M}_0$ *a subfactor, then the possible values of the index* $[\mathbf{M}_1 : \mathbf{M}_0]$ *are restricted to*:

$$[4, \infty] \cup \{4\cos^2(\pi/p), \ \text{where } p \in \mathbb{N}, p \geq 3\}.$$

Moreover, each real number in the continuous part of the spectrum $[4, \infty]$ *and also in the discrete part* $\{4\cos^2(\pi/p), p \in \mathbb{N}, p \geq 3\}$ *is realized.*

We now sketch the idea of the proof. Jones begins with the type II_1 factor $\mathbf{M}_1$ and the subfactor $\mathbf{M}_0$. There is also a tiny bit of additional structure: In this setting there exists a map $\mathbf{e}_1 : \mathbf{M}_1 \to \mathbf{M}_0$, known as the *conditional expectation* of $\mathbf{M}_1$ on $\mathbf{M}_0$. The map $\mathbf{e}_1$ is a *projection*, i.e. $(\mathbf{e}_1)^2 = \mathbf{e}_1$.

His first step is to prove that the ratio λ is independent of the choice of the Hilbert space $\mathfrak{H}$. This allows him to choose an appropriate $\mathfrak{H}$ so that the algebra $\mathbf{M}_2 = \langle \mathbf{M}_1, \mathbf{e}_1 \rangle$ generated by $\mathbf{M}_1$ and $\mathbf{e}_1$ makes sense. He then investigates $\mathbf{M}_2$ and proves that it is another type II_1 factor, which contains $\mathbf{M}_1$ as a subfactor, moreover the index $|\mathbf{M}_2 : \mathbf{M}_1|$ is equal to the index $|\mathbf{M}_1 : \mathbf{M}_0|$, i.e. to λ. Having in hand another II_1 factor $\mathbf{M}_2$ and its subfactor $\mathbf{M}_1$, there is also a trace on $\mathbf{M}_2$ (which by the uniqueness of the trace) coincides with the trace on $\mathbf{M}_1$ when it is restricted to $\mathbf{M}_1$, and another conditional expectation $\mathbf{e}_2 : \mathbf{M}_2 \to \mathbf{M}_1$. This allows Jones to iterate the construction, to build algebras $\mathbf{M}_1, \mathbf{M}_2, \ldots$ and from them a *tower of algebras*:

$$\mathbf{J}_n(\lambda) = \{\mathbf{1}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}\} \subset \mathbf{M}_n, \qquad n = 1, 2, 3, \ldots$$

ordered by inclusion:

$$\mathbf{J}_0(\lambda) \subset \mathbf{J}_1(\lambda) \subset \mathbf{J}_2(\lambda) \subset \cdots$$

where we now regard λ, the index, as a parameter.

The algebras $\mathbf{J}_n(\lambda)$ turn out to have defining relations:

$$(1) \qquad \mathbf{e}_i\mathbf{e}_k = \mathbf{e}_k\mathbf{e}_i \qquad \text{if } |i-k| \geq 2$$

$$(2) \qquad \mathbf{e}_k\mathbf{e}_i\mathbf{e}_k = (1/\lambda)\mathbf{e}_k \quad \text{if } |i-k| = 1$$

$$(3) \qquad (\mathbf{e}_i)^2 = \mathbf{e}_i.$$

Recall that $\mathbf{M}_n$ supports a unique trace, so $\mathbf{J}_n(\lambda)$ does too. It satisfies the condition:

$$(4)_\lambda \qquad \mathrm{tr}(\mathbf{w}\mathbf{e}_n) = (1/\lambda)\,\mathrm{tr}(\mathbf{w}) \quad \text{if } \mathbf{w} \in \mathbf{J}_n(\lambda).$$

Jones proves that (1), $(2)_\lambda$, (3) and $(4)_\lambda$ suffice to calculate the trace on all of $\mathbf{J}_n(\lambda)$. (This is so because, using (1), $(2)_\lambda$ and (3) repeatedly, every element of $\mathbf{J}_n(\lambda)$ can be expressed as a sum of monomials in which the highest subscripted $\mathbf{e}_k$ occurs exactly once, whence $(4)_\lambda$ applies to get rid of that subscript.) The index turns out to be a Laurent polynomial in λ over the integers. The argument concludes when Jones uses its properties to deduce that such a trace could not exist on an infinite sequence of algebras, unless the restrictions expressed by the theorem held.

(Remark: Condition $(4)_\lambda$ asserts that the trace is a "Markov trace." This condition is a key to the connection with knot theory, and will seem natural to a knot theorist who is familiar with the braid group approach to the classification of knots. It seems remarkable that it should also be both natural and a rich source of structure in the world of Von Neumann algebras, a theme which is developed in detail in this monograph.)

The tower of algebras $\mathbf{J}_n(\lambda)$, its various generalizations, and the "basic construction" which we have just described, were at the center of the mathematics which earned Jones the Fields medal. They are the subject of *Coxeter graphs and towers of algebras*. The tower in the very simple construction we just described has associated to it a Dynkin diagram which describes the inclusions of each $\mathbf{J}_n(\lambda)$ in $\mathbf{J}_{n+1}(\lambda)$. This diagram is of type A_n. More general Dynkin diagrams are obtained if one looks at more general derived towers associated to the pair $\mathbf{N} = \mathbf{M}_1$ and $\mathbf{M} = \mathbf{M}_2$, e.g. one can replace $\mathbf{N}$ by $\mathbf{N}' \cap \mathbf{N}$ and $\mathbf{M}$ by $\mathbf{N}' \cap \mathbf{M}$, where $\mathbf{N}'$ is the commutant of $\mathbf{N}$ in $\mathbf{M}$, i.e. set of all elements of $\mathbf{M}$ which commute with every element in $\mathbf{N}$.

This reviewer works primarily in knot theory. We were motivated to take a close look at *Coxeter graphs and towers of algebras* because we had a need to learn about some of the topics covered in it, and for that purpose we found it to be a truly valuable reference. On the other hand, from our point of view there was a lack of overall motivation. The book will no doubt seem quite different to the "experts." It clearly contains much much more than the key reference [Jo,1]. The authors explain rather carefully that they had two audiences in mind. Perhaps, as they say, it is too early to give a full account of the connections between knot theory and this part of Von Neumann algebras. On balance, from my point of view, I think they did a good job.

REFERENCES

[Jo,1] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[Jo,2] ——, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), 335–388.

JOAN S. BIRMAN
COLUMBIA UNIVERSITY