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THE BURAU REPRESENTATION OF THE BRAID GROUP B_n IS UNFAITHFUL FOR LARGE n

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Two fundamental theorems of classical knot theory, by J. Alexander, are that every knot is a closed braid, and second, that a certain procedure (see [1, 2] for both results) assigns to each knot K a polynomial $\Delta(K)$ in T, which only depends on the topological type of the knot once normalized by a power of T to take a positive value at $T = 0$. With this normalization, take T to be a transcendental number.

Since 1926 [2], it has been known that the function Δ: Knots $\to$ $\mathbb{C}$ is not one-to-one. However, it has not become clear whether the composite

$$\begin{array}{ccc} \text{Braids} & \xrightarrow{\text{closure}} & \text{Knots} \\ & \searrow & \downarrow{\scriptstyle \Delta} \\ & & \mathbb{C} \end{array}$$

which is, up to a constant multiple, a virtual character on the positive braids on n strands and, up to normalization, agrees with the extension to a virtual character on the full braid group B_n, yields a faithful virtual character for all n. This is the natural question if one wishes to know whether Δ is an effective tool for studying braids, or for studying knots viewed as closed braids; and it is equivalent to the question of the faithfulness of the Burau representation. Partial results on the latter question were obtained by Magnus, Magnus-Peluso, Lipshultz, Smythe, and others (see, for instance, Magnus' collected works [3]). The connection between braids and knots is developed in Birman's monograph [4] and the connection with mapping-class groups. Using the connection between braids and knots, V. Jones discovered a more general

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polynomial $V(K)$ and new representations of the braid group [5]. Results of D. Long [6] show that for some irreducible components of the new representations, the faithfulness question is the same as for the Burau representation. We announce that the Burau representation is in general not faithful; and, therefore, the hope rests with the remaining components of Jones' representation.[1] The work announced here has already been quite improved by D. Long and M. Paton. As the new work is in progress, let me just briefly mention another new development, by H. Morton, H. Short, and P. Strickland, that it is now known that the Jones representation is strictly more faithful than the Burau.

Let f be the characteristic function of the upper half-plane $\mathbb{H} \subset \mathbb{C}$. Let log denote the natural logarithm function, a multiple-valued function on $\mathbb{C} - \{0\}$. Also, let $\log_0$ denote the principal branch of log, a single-valued function with a discontinuity along the positive real axis, which agrees with one value of log otherwise and satisfies $\log_0(-1) = i\pi$. For any simple closed curve γ in $\mathbb{C} - \{1, \dots, n\}$, which begins and ends at a basepoint z_0 of minimum real coordinate value, let $w_t(\gamma)$, for $t = 1, 2, \dots, n$, be defined to be the following finite Fourier series in the variable λ, which vanishes if γ can be isotoped off the vertical ray $\mathrm{Re}(z) = t$, $\mathrm{Im}(z) > 0$ by a basepoint-fixing isotopy

(1)

$$w_t(\gamma) = (1/2\pi i)e^{-2\pi i t\lambda} \int_\gamma f(z) e^{\lambda(h(z)-h_0(z))}(d\log(z-t) - d\overline{\log(z-t)})$$

for

$$h(z) = \log(z-1) + \cdots + \log(z-n),$$
$$h_0(z) = \log_0(z-1) + \cdots + \log_0(z-n).$$

For $n = 3$, specializing λ to $1/2$, it is quite easy to see that $|w_t(\gamma)|$ is exactly equal to the minimum number of crossings of γ with the vertical ray under based isotopy if γ encloses an odd number m of punctures (but under free isotopy otherwise). Thus for m odd and $n = 3$, $w_t(\gamma)$ is an effective obstruction to isotoping the based curve γ from the ray $\mathrm{Re}(z) = t$, $\mathrm{Im}(z) > 0$. For general n, λ, and γ we have the following theorem:

1. **Theorem.** (i) *If $w_j(\gamma)$ is an effective obstruction to isotoping γ not to intersect the ray* $\mathrm{Re}(z) = j$, $\mathrm{Im}(z) > 0$, *then the Burau representation of B_n is faithful.*

[1] It is not obvious, though true [9], that unfaithful components do not contribute to the faithfulness of the representation.

(ii) *If* $w_j(\gamma)$ *is not an effective obstruction to isotopy, then the Burau representation of* B_{n+3} *is not faithful.*

In the paper, the result is proved with $n+3$ improved to $n+2$ in part (ii), by imposing more precision about the type of simple closed curves we allow. Also, the result in the paper is stronger in that it is proved for a slightly more general representation called the Gassner representation.

Formula (1) is motivated by a description Shaun Bullett once made, in conversation, of Witten's Wilson line calculation [7]. It also calculates Kohno's monodromy of the Polchhammer integrals [8], so I am writing the paper to rely on Kohno's formulation. Formula (2) can be interpreted as an explicit calculation of the coefficient $w_t(\gamma)$ described in the paper.

2. Example. Consider the simple closed curve γ in Figure 1.

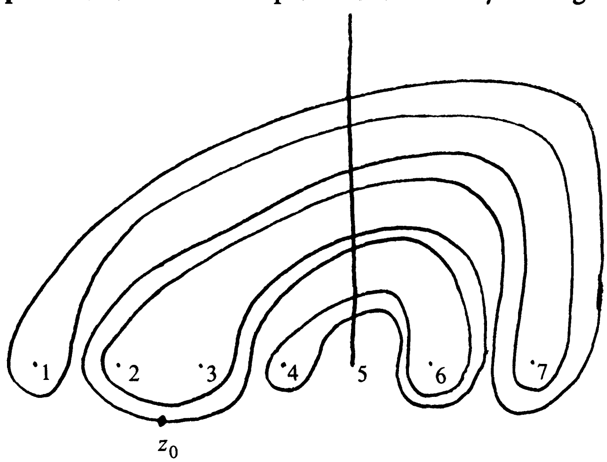

FIGURE 1

Reading around γ, the eight terms of $w_5(\gamma)$ are, for $T = e^{2\pi i\lambda}$,

$$-T^{-2} + T^{-3} - T^{-2} + T^{-1} - T^{-3} + T^{-2} - T^{-1} + T^{-2}$$

These add to zero, yet γ cannot be isotoped off the vertical ray at 5. This shows that the obstruction is not effective for $n = 7$; and, therefore, the Burau representation of B_{10} is unfaithful by (ii).

To illustrate (ii) for the example, consider, not γ, but the related curve η below, and the curve $\partial\eta$ based at z_0, which encircles a small neighborhood of η. Imagine arcs perpendicular to the page, piercing it a 1, 3, 4, 5, 7, and 8. One sees a braid in which the point B, beneath the page, is the starting point of the fourth strand, and A of the sixth. Pulling at A while holding B taught so the arc at 6 slices eight times through γ, has no effect on the sixth coefficient of the Burau image of the fourth basis vector: this contribution is $w_6(\partial\eta) = (1-T)w_5(\gamma) = 0$. Other coefficients are affected in some terms by at most an error of branch number, which can be corrected by passing any other arc through the same eight crossings in the other direction. (See Figure 2.)

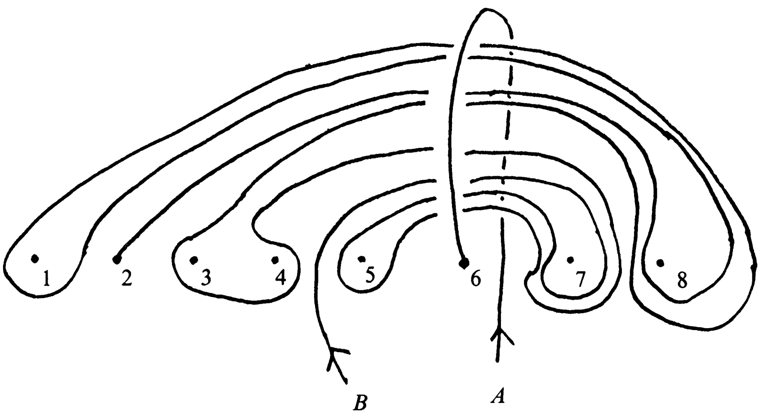

FIGURE 2

The braid in Figure 3 has ten strands, labeled $0, \dots, 9$. It contains four copies of η. We have added to each an additional arc at 6, incrementing 6, 7, and 8, and an arc at 0. Denoting the Burau basis vectors by $e_0, \dots, e_9$, the previous paragraph shows that σ_6 fixes αe_4. The left vertical column, which obviously fixes e_0, then fixes e_1 as well. If the representation were faithful, a simple algebraic argument would show that the left vertical column must then commute with the Burau action of σ_0. Therefore, if the representation is faithful, this braid must lie in the kernel. On the other hand, in order to actually slide the crossing at the bottom to the top, a pair of hands pulling out of the page at A and C would have to bring the complicated arc between to the front of the diagram. This requires isotoping η past the ray $\mathrm{Re}(z) = 6$, $\mathrm{Im}(z) > 0$ (darkened in Figure 3). Equivalently, this requires isotoping γ away from the ray $\mathrm{Re}(z) = 5$, $\mathrm{Im}(z) > 0$. In the example we noted this is impossible. Therefore the braid is not trivial, and the only possibility is that the representation is not faithful.

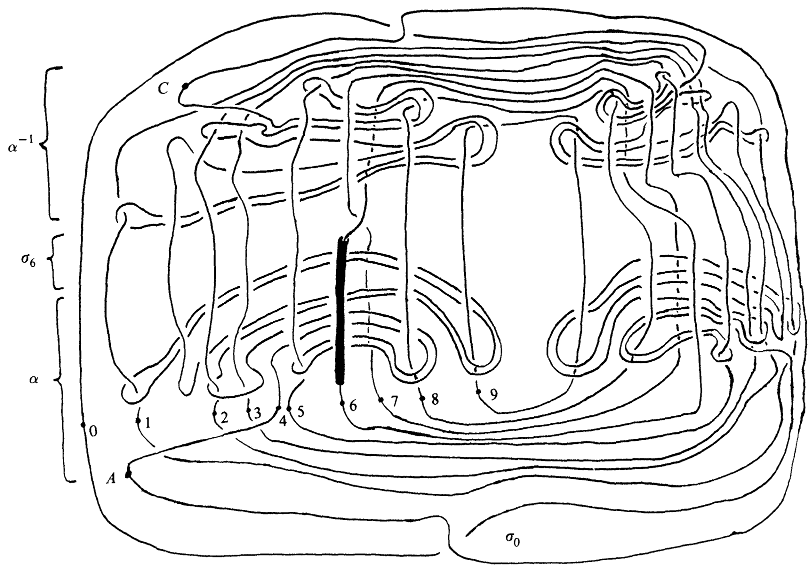

FIGURE 3

I would like to conclude by remarking that I have not found an analogous counterexample for the Gassner representation; while it is still true that the Gassner representation would be faithful just if the appropriate integral

(2)

$$(1/2\pi i)e^{-2\pi i(\lambda_1+\cdots+\lambda_t)} \times \int_\gamma f(z)e^{\sum_j \lambda_j(\log(z-j)-\log_0(z-j))}(d\log(z-t) - d\overline{\log(z-t)})$$

were the only remaining obstruction to isotoping γ off the vertical ray at t.

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