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THE TAIT FLYPING CONJECTURE

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ABSTRACT. We announce a proof of the Tait flyping conjecture; the confirmation of this conjecture renders almost trivial the problem of deciding whether two given alternating link diagrams represent equivalent links. The proof of the conjecture also shows that alternating links have no "hidden" symmetries.

In the nineteenth century, the celebrated physicist and knot tabulator P. G. Tait proposed the following conjecture: given reduced, prime alternating diagrams D_1, D_2 of a knot (or link), it is possible to transform D_1 to D_2 by a sequence of *flypes*, where a flype is a transformation most easily described by the pictures of Figure 1 on p. 404.

In performing a flype, the tangle represented by the shaded disc labelled S_A is turned upside-down so that the crossing to its left is removed by untwisting, and a new crossing is created to its right; if the tangle diagram S_A has no crossing, the flype leaves the link diagram unchanged up to isomorphism, whereas if the tangle diagram S_B should have no crossing, the flype amounts merely to a rotation of the complete link diagram about an axis in the projection 2-sphere. During the last few years, some partial results have appeared; in particular it follows from the analysis of [B-S] on arborescent links that any two alternating diagrams of a link which are *algebraic* (i.e. which have Conway basic polyhedron 1*) must be related via a sequence of flypes. A slightly stronger version of this result is set forth in [T4], where the conclusion is obtained for a pair of alternating diagrams only one of which is given as algebraic. It follows from the results of [B-M] that the Tait conjecture holds for link diagrams which are closures of alternating 3-string braid diagrams. K. Murasugi and J. Przytycki [M-P] have proved a number of results on graph polynomials which have lent support to the conjecture. Very recently, A. Schrijver [S] has announced

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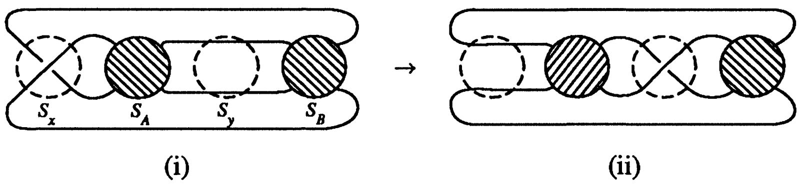


FIGURE 1

a proof for those alternating diagrams which do not admit any decomposition into two nontrivial tangle diagrams.

The purpose of this article is to announce a proof of a strengthened form of Tait's conjecture, set forth as Theorem 1 below; by considering a flype as being a certain kind of homeomorphism of (3-sphere, link)-pairs, rather than a mere transformation of diagrams, we are able to gain information on symmetries of alternating links. The proof of Theorem 1 is mostly elementary: it stems from three sources, namely (i) work of the first author on 4-punctured 2-spheres in alternating link complements [M], (ii) work of the second author on properties of the Jones and Kauffman polynomials [T1–T4], and (iii) techniques developed jointly by the authors for handling incompressible surfaces with nonmeridional boundary in alternating link exteriors [M-T1]. The incompressible surfaces exploited here are the two spanning surfaces arising naturally from a checkerboard coloring of the link diagram (see for instance [G]). Since we consider the complex which is the union of these two surfaces, the analysis is considerably more intricate than that of [M-T1]. The most basic use of polynomials is in ensuring that any two reduced alternating diagrams of a given link have the same number of crossings, but more subtle use of polynomials is made at various other stages of the proof. Thus, although the proof of Theorem 1 has a strong geometric flavor, it is not entirely geometric; the question remains open as to whether there exist purely geometric proofs of this and other results which have been obtained with the help of the new polynomial invariants. Full details of the proof of Theorem 1 are given in [M-T2].

As in [M-T1], for each n-crossing diagram D of a link L, we take small "crossing-ball" neighborhoods $B_1, \dots, B_n$ of the crossing-points of D, and then assume without loss of generality that L coincides with D, except that inside each B_i the two arcs of $D \cap B_i$ are perturbed vertically to form semicircular overcrossing and undercrossing arcs which lie on the boundary of B_i. We express this relationship between the link L and its diagram D by writing $L = \lambda(D)$.

We may assume that the B_i are Euclidean balls of fixed (unit) radius, and that each pair $(B_i, B_i \cap \lambda(D))$ is isometric to a "standard pair" (B, L), where B is the unit ball in $\mathbb{R}^3$ and $L = \{(x, y, z) \in \partial B \mid (x = 0 \text{ and } z \geq 0) \text{ or } (y = 0 \text{ and } z \leq 0)\}$.

It is convenient to introduce the notion of a *triple* $\mathcal{T} = (D_1, D_2, f)$; the entries D_1, D_2 in $\mathcal{T}$ are oriented, alternating link diagrams, and f is an orientation-preserving homeomorphism of pairs $(S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$. We consider that such homeomorphisms f_0, f_1 are equivalent if they are connected by an isotopy of pairs $f_t : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$ $(0 \leq t \leq 1)$. Thus, if $D_1 = D_2$, an equivalence class of homeomorphisms is simply an element of the mapping class group of the pair of oriented spaces $(S^3, \lambda(D_1))$.

In order to give a precise definition of "flype", we need to consider a certain "trivial" kind of homeomorphism of link pairs.

Definition. A homeomorphism $g : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$ is *flat* if it is pairwise isotopic to a homeomorphism h for which there exists a product neighborhood $N = S^2 \times [-1, 1]$ of the projection 2-sphere S^2, containing both links $\lambda(D_1), \lambda(D_2)$, such that h maps N onto itself and $h = h_0 \times \mathrm{id}_{[-1,1]}$ for some orientation-preserving homeomorphism $h_0 : S^2 \to S^2$.

If we regard S^3 as being the suspension of the projection 2-sphere S^2, then any flat homeomorphism is pairwise isotopic to the suspension of an orientation-preserving autohomeomorphism of S^2. Note that a composite of flat homeomorphisms is also flat; also, since the complement of N in S^3 consists of two disjoint open 3-balls, it is clear that any two flat homeomorphisms between given pairs $(S^3, \lambda(D_1))$ and $(S^3, \lambda(D_2))$ must be pairwise isotopic.

We define the term *flype* in two stages. First, let us consider a diagram D_1 conforming to the "standard" pattern of Figure 1(i), where the circles illustrated in that figure represent Euclidean 2-spheres meeting the (extended) projection plane in their respective equators. Thus the 2-spheres S_A, S_B enclose tangles which may or may not have crossings, S_x is the boundary of a crossing-ball for the crossing x, and S_y bounds a ball meeting the link $\lambda(D_1)$ in two parallel straight line segments.

Definition. Let D_1 be a diagram with standard pattern as in the previous paragraph. A *standard flype* of $(S^3, \lambda(D_1))$ for that pattern is any homeomorphism f which maps $(S^3, \lambda(D_1))$ to a pair $(S^3, \lambda(D_2))$ where D_2 conforms to the pattern of Figure 1(ii), in such a way that (i) f maps the tangle bounded by S_A into itself by a rigid rotation through π about an axis in the projection plane, (ii) f fixes pointwise the tangle bounded by S_B, (iii) f maps each of the tangles bounded by S_x, S_y into itself by a half-twist.

It is easy to see that if f_1, f_2 are two standard flypes of a pair $(S^3, \lambda(D_1))$, with given standard pattern for D_1, then $f_2^{-1} \circ f_1$ is pairwise isotopic to a homeomorphism of $(S^3, \lambda(D_1))$ to itself which fixes pointwise all crossing-balls of D_1, and which maps each complementary region of D_1 into itself; therefore $f_2^{-1} \circ f_1$ is pairwise isotopic to the identity, and so f_1 is pairwise isotopic to f_2.

In the following generalized definition, the flat homeomorphisms g_1, g_2 constitute "choices of coordinates" for the link $\lambda(D_1)$ and its image under the flype.

Definition. Let D_1 be any diagram. Then a *flype* is any homeomorphism $f : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$ of form $f = g_1 \circ f' \circ g_2$ where f' is a standard flype and g_1, g_2 are flat.

We can now state the main result.

Theorem 1. *Let D_1, D_2 be reduced, prime, oriented, alternating link diagrams, and let there be an orientation-preserving homeomorphism of pairs $f : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$. Then f is pairwise isotopic to a composite of flypes.*

It follows at once from this Theorem that if D satisfies the conditions of the hypothesis, the link $\lambda(D)$ can only be amphicheiral if the diagram D can be flyped to its mirror-image in the projection plane, and that $\lambda(D)$ can only be reversible if the diagram D can be flyped to its reverse. Thus we now have a trivial criterion for determining whether an alternating diagram presents an amphicheiral or reversible link. Also, it follows that *any* element of the mapping class group of the pair $(S^3, \lambda(D))$ must be "obvious", in the sense that it must arise from flypes and symmetries of the planar graph underlying the diagram D. In particular, if the graph underlying D has no nontrivial symmetry and D does not admit any nontrivial flype, then any orientation-preserving homeomorphism of $(S^3, \lambda(D))$ to itself is pairwise isotopic to the identity.

The broad plan is to prove Theorem 1 by induction on the number of crossings of D, but to obtain an inductive argument it is necessary to enlarge the scope of the Theorem to the more general category of 4-valent rigid-vertex graphs; one can think of the vertices of such a graph as flat square-shaped disks which are mapped isometrically by morphisms of the category. We assume that in any diagram of such a graph the vertices lie on the projection 2-sphere, and we say that such a diagram is *alternating* if it is possible to obtain an alternating link diagram by substituting a single crossing for each graph vertex. An orientation of a graph consists simply of an orientation of each edge of the graph, and a link is simply a rigid-vertex graph with no vertices. As with link diagrams, a diagram of a graph is said to be *reduced* if it contains no nugatory crossing.

Many of the standard rigidity results for reduced alternating link diagrams (e.g. invariance of crossing-number) may easily be extended to alternating 4-valent rigid-vertex graph diagrams, by means of the device of substituting either single crossings or appropriate tangle diagrams for the rigid vertices. Ironically, substitution of the 5-crossing tangle diagram  for each graph vertex will instantly prove Theorem 1 for alternating rigid-vertex graph diagrams, given the conclusion of Theorem 1 for alternating link diagrams; however, this is not inconsistent with the above inductive procedure, as the deduction of Theorem 1 for a graph diagram with n crossings and v vertices requires the knowledge that Theorem 1 holds for link diagrams with more than n crossings (namely $n + 5v$ crossings).

The notion of flype extends naturally to rigid-vertex graph diagrams with at least one crossing; in the case where D_1, D_2 are rigid-vertex graph diagrams with no crossing, we have $\lambda(D_i) = D_i$ $(i = 1, 2)$, and we say that a *flype* of $(S^3, \lambda(D_1))$ to $(S^3, \lambda(D_2))$ is simply a homeomorphism between those pairs which maps the projection 2-sphere S^2 to itself. Given such diagrams D_1, D_2, let $f : (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$ be any orientation-preserving homeomorphism. Then by Lemma 1 of [D-T] the embedding of $\lambda(D_1)$ is uniquely determined up to homeomorphism of S^2, so we can pairwise isotope f so that it maps each region of D_1 in S^2 to a region of D_2 in S^2; thus f is a flype according to the above definition, and the conclusion of Theorem 1 holds in this special case.

The basis of the induction is therefore established. The proof contains three distinct methods for achieving the inductive step, to which we shall refer as Inductive Arguments A, B, and C. Each of these arguments uses polynomial techniques; the simplest, namely Inductive Argument B, applies whenever there is a simple closed curve in one of the diagrams D_1, D_2 meeting a vertex and a crossing, in the manner of Figure 3(i) on p. 410. Inductive Argument C applies whenever one of the pairs $(S^3, \lambda(D_i))$ admits an essential Conway sphere which is "visible" in the sense that it meets the projection 2-sphere in a circle separating the diagram into two tangle diagrams; it is shown by means of a delicate geometric argument that Inductive Argument A applies in all other cases.

There now follows a brief description of Inductive Arguments A and B. Before outlining Inductive Argument A, we need to describe the "black" and "white" spanning surfaces, which consititute the main tool for keeping track of maps of link or graph pairs. If we color the complementary regions of a link or graph diagram D alternately black and white, then there is a spanning surface β which coincides with the black regions outside the crossing-balls B_i, and which intersects each B_i in a "twisted rectangle." In the case of a graph, we exclude from β the interiors of the graph vertices. In similar fashion, the white regions of D give rise to a spanning surface ω; β, ω are the *black* and *white* spanning surfaces respectively for $\lambda(D)$, associated with the diagram D. In the case of a knot, if such a surface should happen to be orientable, it is simply the Seifert surface obtained from D by Seifert's algorithm. An example is illustrated in Figure 2(ii), (iii).

Since the ambient space S^3 is equipped with an orientation, we may unambiguously decree which complementary regions of D are black, and which are white, by means of the convention illustrated in Figure 2(i). Of course, this convention also decrees which spanning surface is black, and which is white. The following easy result explains partly why the surfaces β, ω are useful in achieving the inductive step.

Proposition 2. *Let* $\mathcal{T} = (D_1, D_2, f)$ *be a triple, and let* β_i, ω_i *be the black and white spanning surfaces respectively for* D_i, $i = 1, 2$; *also, for each* i *let* B_i *be a crossing-ball for* D_i. *If the homeomorphism* $f\colon (S^3, \lambda(D_1)) \to (S^3, \lambda(D_2))$ *maps* $(B_1, B_1 \cap$

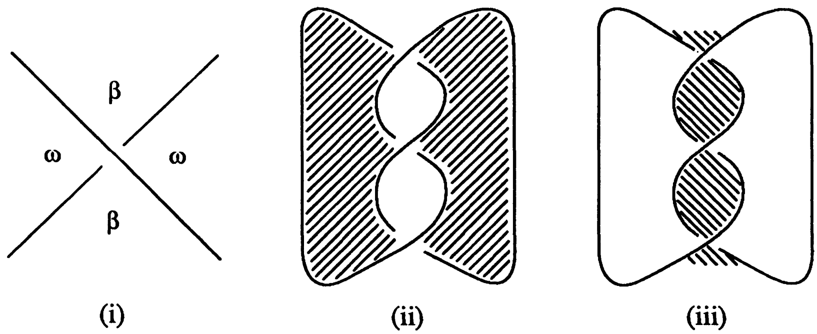


FIGURE 2. (i); (ii) β =punctured torus;
(iii) ω =Mobius band.

β_1) onto $(B_2, B_2 \cap \beta_2)$, or $(B_1, B_1 \cap \omega_1)$ onto $(B_2, B_2 \cap \omega_2)$ then f is pairwise isotopic to a homeomorphism mapping B_1 isometrically to B_2.

As already mentioned, in the proof of Theorem 1 it is necessary to look at both black and white surfaces simultaneously; note that they intersect transversely in arcs which we may assume to be the vertical "polar axes" of the crossing-balls. Also, note that the Euler characteristic of a black or white spanning surface is equal to the number of regions of the color of that surface minus the number of crossings of the diagram; therefore the sum of the Euler characteristics of the two spanning surfaces is equal to $2 - c + v$, where c, v are the numbers of crossings and graph vertices of the diagram, respectively. This last fact is of crucial importance in the proof that the hypotheses of at least one of the three inductive arguments is fulfilled.

Suppose we are given a triple $\mathscr{T} = (D_1, D_2, f)$; let β_i, ω_i be the associated black and white spanning surfaces. Then in the exterior of $\lambda(D_2)$ we have four surfaces $f(\beta_1), f(\omega_1), \beta_2, \omega_2$. An essential preliminary step in the proof of Theorem 1 is to show that the homeomorphism f can be pairwise isotoped so that these four surfaces intersect transversely in a "nice" manner, similar to "standard position" in [M-T1]. As a very first step, it is necessary to show that the surfaces are incompressible. This, however, is a routine exercise; to show incompressibility of β, for instance, one examines the (transverse) intersection of a purported compressing disk with ω.

Definition. Let F be one of $f(\beta_1)$, $f(\omega_1)$, and let G be one of β_2, ω_2, considered as surfaces properly embedded in the exterior X of $\lambda(D_2)$. A double arc α of $F \cap G$ is *excellent* if

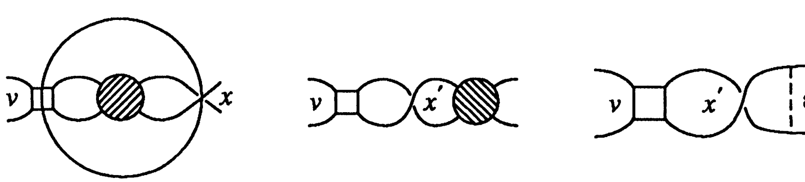
(i) α contains no triple point;
(ii) there is a neighborhood N of α in X such that the quadruple $(N, N \cap \partial X, N \cap F, N \cap G)$ is homeomorphic to $(B \cap X, B \cap \partial X, B \cap X \cap \beta, B \cap X \cap \omega)$, where B is a crossing-ball for D_2.

Thus the two surfaces F, G intersect at α in the same way that the black and white spanning surfaces intersect at the polar axis of a crossing-ball.

We may produce two new links (or graphs) by cutting F, G respectively along α, and taking the boundary of each cut surface. By comparing polynomials of these links and examining nugatory crossings, we can establish Inductive Argument A, which asserts that up to alteration of f by flypes, the existence of an excellent arc guarantees that f maps some crossing-ball for D_1 rigidly to some crossing-ball for D_2; we can then replace these crossing-balls by rigid vertices, thus reducing the number of crossings. The hardest part of the proof of Theorem 1 is in producing an excellent arc in the case where Inductive Arguments B, C do not apply.

We conclude with an outline of Inductive Argument B. As explained above, this argument deals with the situation illustrated in Figure 3(i), which we may assume to occur in the diagram D_1 without loss of generality.

Inductive Argument B then asserts that, after a possible flype, we can find a crossing-ball of D_1 mapped rigidly to a crossing-ball of D_2. As with Inductive Argument A, we can then replace these crossing-balls with graph vertices, thus reducing crossing-number.



(i) (ii) (iii)

FIGURE 3

Sketch proof of Inductive Argument B. First, observe that if we replace in Figure 3(i) the vertex v by the vignette, the crossing x becomes nugatory; therefore, by rigidity of vertices and invariance of nugatory crossings, the corresponding replacement of the vertex $f(v)$ by the same vignette in D_2 must also render some crossing y say of D_2 nugatory. But then there must be a circle in the projection plane meeting $f(v)$ and y in the same way that C meets v and x. Now it is easy to see that there are flypes of the diagrams which "transfer" the crossings x, y to crossings x', y' immediately adjacent to the graph vertices $v, f(v)$ respectively (Figure 3(ii)). A minor technicality arises at this stage; it is necessary to show that the crossings x', y' are twisted in the same sense in relation to $v, f(v)$ respectively, but this is fairly easy to accomplish. Let us suppose that the crossings x', y' are such that there are black regions between the crossings and the vertices, as illustrated in Figure 3(ii). It is then a routine matter to show that f can be pairwise isotoped so that the "rectangular" part of the black spanning surface for D_1, between v and the arc ϵ illustrated in Figure 3(iii), is mapped onto a rectangular part of the black spanning surface for D_2 which is similarly positioned in relation to $f(v)$. By adjusting f further if necessary, we may assume that the hypothesis of Proposition 2 is satisfied for the crossings x', y'. It then follows from Proposition 2 that f is pairwise isotopic to a homeomorphism mapping the crossing-ball at x' rigidly to the crossing-ball at y'.

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