BOOK REVIEWS

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Vertex operator algebras and the Monster, by Igor Frenkel, James Lepowsky, and Arne Meurman. Academic Press, New York, 1988, 502 pp., $69.95. ISBN 0-12-276065-5

The classification of all finite simple groups was achieved about ten years ago, the fruit of many years of work by scores of mathematicians and requiring thousands of journal pages for the proof. Besides the alternating groups and sixteen infinite families of groups of Lie type, there are twenty-six sporadic groups, the last and most glamorous being the *Monster* M, of order

$$|M| = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \sim 8\cdot 10^{53}.$$

Its existence was conjectured in 1973 by Fischer and by Griess, together with a nontrivial rational representation of (minimal) degree

$$d_2 = 196883 = 47 \cdot 59 \cdot 71.$$

For a period of about eight years, the Monster, like its namesake in Loch Ness, was not known to exist, but enough observations had been made to support a considerable body of theory. In particular, Fischer, Livingstone, and Thorne had computed the character table, and Norton knew that the representation space of dimension 196883 must be an algebra of a certain kind.

On the other hand, we have modular functions, a subject which has been flourishing for nearly two centuries, and doing especially well in recent years; it has always been central to number theory and with close ties to many parts of mathematics. For example, the *elliptic modular invariant* $j(\tau)$ classifies the isomorphism classes of elliptic curves over $\mathbb{C}$, and has the Fourier expansion

$$j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n,$$

where $q = e^{2\pi i\tau}$ and τ is in the upper half-plane. The coefficients $c(n)$ are positive integers, growing rapidly, with

$$c(1) = 196884.$$

The function $j(\tau)$ is a "Hauptmodul" for the modular curve X, which is the upper half-plane divided by the modular group $\mathrm{PSL}(2, \mathbb{Z})$, completed by adding the cusp ∞; thus $j(\tau)$ is an isomorphism of X onto the Riemann sphere, normalized to have the leading term q^{-1} at ∞; it is unique up to the constant term $c(0)$. We are also concerned with modular curves of higher level, especially the curves $X_0(N)$, which parametrize pairs (E, C), where E is an elliptic curve and C is a cyclic subgroup of order N, corresponding to the subgroup of the modular group defined by the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with c divisible by N, and the quotients

$$X_0^+(N) = X_0(N)/W$$

by a group of involutions W (or order 2^r if r distinct primes divide N). The first indication that something strange was going on was the reviewer's light-hearted observation in 1975 that the fifteen primes which divide $|M|$ are exactly those primes p for which $X_0^+(p)$ has genus 0, or, equivalently, those for which the Hasse polynomial splits to linear factors over the field with p elements. Soon after, McKay made the striking observation that

$$196884 = 196883 + 1,$$

or $c(1) = d_2 + d_1$, $d_1 = 1$ being the degree of the trivial representation, and he and Thompson then found $c(2) = d_1 + d_2 + d_3$, and similar linear relations for the first few values of n, where d_k is the degree of the kth irreducible rational representation of M. By 1979 Conway and Norton, in their amazing paper *Monstrous moonshine*, had shown (with help form Thompson, Atkin, Fong, and Smith) that there is a module V_n of dimension $c(n)$ on which M acts, for $n \geq 1$, i.e. a graded M-module

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots$$

(V_{-1} has dimension 1), for which the formal series

$$\dim V = \sum_n (\dim V_n) q^n = q^{-1} + c(1)q + c(2)q^2 + \cdots$$

is equal to $j(\tau) - 744$. Furthermore, if we let χ_n be the corresponding character of M (so $\chi_n(g)$ is the trace of the action of $g \in M$ on V_n), then the series

$$j(g, \tau) = q^{-1} + \sum_{n=1}^{\infty} \chi_n(g) q^n$$

is a Hauptmodul for a factor of some $X_0(N)$, of genus 0, with the level N related to the order of g, e.g. if g has order 71 (g is unique up to conjugacy), then $j(g, \tau)$ is the Hauptmodul for $X_0^+(71)$.

Over the same period of time (the last twenty years or so), the theory of infinite dimensional Lie algebras and their relations to modular functions has made great strides, starting with Macdonald's systematic derivation of theta identities by means of affine root systems. Here one considers infinite dimensional graded Lie algebras with finite dimensional summands, not necessarily of Kac-Moody type; if $V = \bigoplus_n V_n$ is a graded module for such a graded Lie algebra, then its formal dimension

$$\dim V = \sum_n (\dim V_n) q^n$$

(assuming each $\dim V_n$ to be finite) might be a modular function of some kind. For example, $j^{1/3}$ corresponds to such a module for the affine Lie algebra $\widetilde{E}_8$, thereby "explaining" $744 = 3 \cdot (\dim E_8)$.

Now the discoveries of Conway and Norton had suggested that the most natural representation of M was a graded infinite dimensional one, and so Lie theorists, including Kac and the authors of the book under review, thought that their methods should help to explain moonshine (as the new field was now called). This view met with skepticism at first but was vindicated by later developments.

Griess constructed the Monster about 1981, in a work of striking power and originality. He constructed the group with an irreducible representation on a space of dimension 196883 and made a 1-dimensional extension V_1 (of dimension $c(1) = 196884$) into a commutative algebra with identity and with an invariant form, as Norton had predicted. In the book under review, the authors (with important contributions by Borcherds) show that there is a graded module V on which the Monster acts, with $\dim V = j(\ ) - 744$ (i.e., $\dim V_n = c(n)$), which is a vertex algebra (defined below) with component V_1 equal to the Griess algebra. So the structure of algebra was extended, in a certain sense, to the natural infinite dimensional "moonshine module."

A *vertex* (*operator*) *algebra*, over a field $\mathbb{F}$ of characteristic 0, consists of

(a) a vector space V,
(b) a linear map $V \to (\operatorname{End} V)[[z, z^{-1}]]$, written $u \mapsto Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-(n+1)}$, and
(c) a *vacuum vector* $\mathbf{1} \in V$,

subject to axioms (I) through (V) below. Note that the nth linear map $u \mapsto u_n$ of V into End V defines a structure of *algebra* on V, i.e. a bilinear map $V \times V \to V$ by $(u, v) \mapsto u_n \cdot v$, so V is an algebra in infinitely many ways, indexed by $\mathbb{Z}$. The treatment in the book is entirely algebraic; z is a formal variable, and there is no notion of convergence in $\mathbb{F}$ or in V, so a sum in a vector space must be essentially finite to be meaningful. (A sum $\sum_i f_i$ in End V can be infinite, but $\sum_i f_i \cdot u$ must be finite for any $u \in V$.) The axioms are:

(I) If $u, v \in V$, then $u_n \cdot v = 0$ for n large enough (depending on u and v). Thus $Y(u, z)\cdot v$ is meromorphic in z, although $Y(u, z)$ need not be.
(II) $Y(\mathbf{1}, z) = I$ (the identity in End V), i.e. $\mathbf{1}_n$ is I for $n = -1$ and 0 for $n \neq 1$.
(III) $Y(u, z) \cdot \mathbf{1}|_{z=0} = u$, i.e. $u_n \cdot \mathbf{1}$ is 0 for $n \geq 0$ and is u for $n = -1$. Thus the *vertex operator* $Y(u, z)$ creates the state u out of the vacuum.
(IV) The *Jacobi identity*: for $u, v \in V$, and independent formal variables z_0, z_1, z_2, we have

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y(v, z_2)Y(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2).$$

The δ-functions are expanded binomially in nonnegative powers of the right-hand variable, so the first two δ-functions are not the same, although they look the same. This is a triply indexed family of identities; for example, equating coefficients of $z_0^{-1} z_1^{-(m+1)} z_2^{-(n+1)}$ gives the commutator formula

$$[u_m, v_n] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i \cdot v)_{m+n-i}$$

(a finite sum, by (I)).

Before stating the fifth axiom, let us make a few remarks about what we have so far. There is a *canonical derivation* $D: V \to V$ defined by $Du = u_{-2} \cdot \mathbf{1}$, satisfying

$$Y(Du, z) = d/dz Y(u, z) = [D, Y(u, z)].$$

The third axiom can be extended to

$$Y(u, z) \cdot \mathbf{1} = e^{zD} u.$$

It is easy to check that a vertex algebra is commutative if and only if it is integral (i.e. $u_n = 0$ for all $u \in V$ and all $n \geq 0$), in which case it is just a commutative ring (associative algebra with identity) with a derivation D. In the general case, you can think of a vertex algebra as a generalized ring, with identity element $\mathbf{1}$ and derivation D. No really easy noncommutative examples are known; the first half of the book constructs a vertex algebra starting from any even nondegenerate lattice L, and the second half constructs a more exotic one on which the Monster acts.

The last axiom is:

(V) There is a *conformal vector* $\omega \in V$ of *rank* r such that
  (i) $\omega_0 = D$;
  (ii) ω_1 defines a grading $V = \bigoplus V^k$, where V^k is of finite dimension and is 0 for k sufficiently small; if $u \in V^k$, then $\omega_1 u = k \cdot u$ and we say that u has *weight* equal to k;
  (iii) the operators $L(n) = \omega_{n+1}$ satisfy the Virasoro relations

$$[L(m), L(n)] = (m-n)L(m+n) + (m^3 - m) \cdot \delta_{m,-n} \cdot r/12.$$

The first half of the book establishes a natural structure of vertex algebra on a vector space V_L attached to an even lattice L; we will describe the untwisted case here and not say much about the twisted case, which is equally important. By definition, L is a free $\mathbb{Z}$-module of rank r together with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on L or on $\mathfrak{h} = L \otimes \mathbb{F}$ (regarded as an abelian Lie algebra of dimension r), with $\langle \alpha, \alpha \rangle \in 2 \cdot \mathbb{Z}$ whenever $\alpha \in L$. Then

$$V_L = \bigoplus_{\alpha \in L} V \otimes e^{\alpha},$$

where e^{α} is a formal exponential, and V is a polynomial algebra $\mathbb{F}[\ldots, x_i(n), \ldots]$ in infinitely many variables $x_i(n)$, for $1 \leq i \leq r$, and $n = 1, 2, \ldots$. The space V_L is graded by assigning to the basis element $u = x_{i_1}(n_1) \cdots x_{i_k}(n_k) \cdot e^{\alpha}$ the weight

$$w(u) = n_1 + \cdots + n_k + \langle \alpha, \alpha \rangle /2.$$

If L is positive (i.e. $\langle \alpha, \alpha \rangle > 0$ whenever $\alpha \neq 0$), then the weight spaces are finite dimensional and are 0 for negative weights, and so after multiplying by $q^{-r/24}$ (the "degree shift"), we find

$$\dim V_L = q^{-r/24} \sum_w (\dim V^w) q^w = \Theta_L(q)/\eta(q)^r$$

where $\eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's function and $\Theta_L(q) = \sum_{\alpha \in L} q^{\langle \alpha, \alpha \rangle /2}$ is the theta function of the (positive even) lattice L. Thus $\dim V_L$ is a modular function of some level. The Heisenberg algebra

$$\mathbb{F} \cdot \mathbf{1} \oplus \bigoplus_{n \neq 0} \mathfrak{h} \otimes t^n$$

(t being a formal symbol) acts on V in a natural way. (Let $\alpha_1, \ldots, \alpha_r$ be a basis of $\mathfrak{h}$, and let $\alpha_1', \ldots, \alpha_r'$ be the dual basis relative to the form $\langle\ ,\ \rangle$. Then $\alpha_i(n) = \alpha_i \otimes t^n$ acts as $\frac{\partial}{\partial x_i(n)}$, for $n < 0$, and $\alpha_i'(-n)$ acts as $n \cdot x_i(n)$.) The *current* defined by $\alpha \in \mathfrak{h}$ is

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-(n+1)},$$

where the action of $\alpha(0)$ on V_L is suitably defined. Then, very roughly, the vertex operator $Y(u, z)$ of a "constant" vector $u = e^{\alpha}$ is

$$\substack{\circ\\ \circ} e^{\int \alpha(z) dz/z} \substack{\circ\\ \circ}$$

where the open colons denote *normal ordering*, in which the operators indexed by $n < 0$ are placed to the left of those indexed by $n > 0$ (and we ignore here the considerable work required to deal with the case $n = 0$). This notion of normal ordering is required in order to make the operator well defined (sums must be finite) and has many pleasant algebraic consequences; it is used frequently throughout the book. The definition is extended to general $u \in V_L$ so that replacing u by $\beta(-m) \cdot u$ replaces $Y(u, z)$ by

$$\substack{\circ\\ \circ}(1/(m-1)!)(d/dz)^{m-1} \beta(z) Y(u, z)\substack{\circ\\ \circ}.$$

There is a natural conformal vector $\omega \in V_L$, of rank r, and this makes V_L into a vertex algebra.

Thus, if L is a positive even lattice, then we have a vertex algebra V_L for which $\dim V_L$ is a modular function of some level. If L is unimodular, then $\dim V_L$ is of level 1 (invariant under the full modular group $\mathrm{PSL}(2, \mathbb{Z})$). In particular, if we take the *Leech lattice* Λ, unimodular of rank 24 and with no short vectors (no $\alpha \in \Lambda$ satisfies $\langle \alpha, \alpha \rangle = 2$), then we have

$$\dim V_\Lambda = j(\tau) - 720 = q^{-1} + 24 + 196884q + \cdots .$$

This is not yet the right ingredient for making moonshine, as the constant term of 24 reveals; it turns out that the correct *moonshine module* is

$$V^\natural = V_\Lambda^+ + (V_\Lambda')^+ ,$$

where V_Λ^+ is the subalgebra of V_Λ fixed by a certain involution, and $(V_\Lambda')^+$ is the analogous twisted object, which is a module (in a suitable sense) for V_Λ^+. Furthermore, there is a large group C of automorphisms of $V^\natural$, preserving the two summands; the group C is an extension of a Conway group by an extraspecial 2-group and is isomorphic to the centralizer of an involution in M. It is easy to check that $\dim V^\natural = j - 744$, with constant term 0, which is necessary and highly encouraging.

In the last third of the book, which is naturally the most difficult part, the moonshine module is given the structure of a vertex algebra. This involves the action of a permutation group S_3 on three letters ("principle of triality"), mixing up the twisted and untwisted parts. It is proved that M is the full automorphism group of $V^\natural$, and is generated by C and S_3. The weight 2 subspace is the Griess algebra, and the conformal vector ω is the identity in the Griess algebra.

There have been interesting developments in physics, in string theory, and related subjects, also during the last twenty years or so, and the mathematical side has been greatly enriched by ideas from physics. Physicists have been coming into contact with modular functions for some time, and the role of modular functions seems to be essential, if quite mysterious; physicists have also made considerable use of infinite dimensional Lie algebras and contributed to the development of that theory. It seems that the theory of vertex algebras is equivalent to 2-dimensional conformal field theory; the Monster is the symmetry group of a particular conformal field theory and would have been discovered by physicists if the group theorists had not found it first.

The reader will have surmised by now that the reviewer has a very high opinion of this book. The exposition is not perfect, with some repetition of arguments, and too many formulas, but such quibbles are overwhelmed by the service the authors have rendered by making such fascinating material available to a wide audience. The authors needed five years to write up their results; an outsider trying to learn all of this from partial results in the literature would have faced an impossible task. The book is accessible to any motivated reader, although it is not easy reading; it is essentially self-contained, except for a few results which are cited (with references) in the later chapters. It cannot be the definitive treatment, since the subject is just getting started; recent developments include Borcherds' "two-variable moonshine," with many new concepts and results, but it is impossible to imagine a future treatment of these matters which would not be heavily influenced by this book.

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Group theory in physics, volume III, *Supersymmetries and infinite-dimensional algebras*, by J. F. Cornwell. Techniques of Physics (N. H. Marsh, ed.), Academic Press, New York, 1989, 615 pp., $55.00. ISBN 0-12-189805-9

The theory of classical Lie groups and algebras is of fundamental importance in mathematics as the meeting ground of algebra, analysis, and topology, and as an essential tool in modern physics. The book under review is the third in a series of volumes by Cornwell. In the first volume he covers some basic group theory, Lie groups, representation theory, and applications to molecular and solid state physics. In the second volume he discusses Lie algebras, their relationship with Lie groups, the structure theory of semisimple Lie algebras and their representation theory, Lorentz groups, Poincaré groups, and applications to the theory of elementary particles (global internal symmetries and gauge theory). Results and numbered equations from these two volumes are referred to in