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*Hilbert modular forms*, by Eberhard Freitag. Springer-Verlag, Berlin and New York, 1990, 250 pp., $49.00. ISBN 3-540-50586-5

The Hilbert modular group is a straightforward generalization of the classical modular group $SL(2, \mathbf{Z})$ and is obtained by replacing the ring of rational integers by the ring of integers of a totally real algebraic number field. Let $K$ be the field in question and $O_K$ its ring of integers. The action of the classical modular group on the upper half-plane $H$ in the complex numbers (via fractional linear transformations) generalizes to an action of the Hilbert modular group $SL(2, O_K)$ on $H^n$ via the $n = [K : \mathbf{Q}]$ embeddings of the field $K$ into the real numbers. Hilbert himself explained in his famous turn-of-the-century lecture at the International Congress in Paris why he was interested in this generalization [5]. His starting point was the well-known theorem of Kronecker(-Weber) which says that any number field which is abelian over $\mathbf{Q}$ is contained in a cyclotomic number field $\mathbf{Q}(\zeta_m)$ and the beautiful (though at that moment partly conjectural) extension of this to imaginary quadratic number fields by the theory of complex multiplication. Here Hilbert followed Kronecker who had expressed as his philosophy (his "Jugendtraum," cf. a letter to Dedekind of 1880 in *Werke*. V) that every abelian extension of an imaginary quadratic number field $L$ is contained in a number field obtained by adjoining special values of elliptic and modular functions. Hilbert wanted to find the analytic functions which play the same role for arbitrary algebraic number fields as the exponential function does in the Kronecker-Weber theorem and the $j$-function does in the theory of complex multiplication. ("Ich halte dies Problem für eines der tiefgehendsten und weittragendsten Probleme der Zahlen und Funktionentheorie.") Hilbert aimed at nothing less than a theory of modular functions of several variables which should be as important in number theory and geometry as the theory of modular functions was at the end of the last century.

The group itself had appeared earlier in work of Humbert, when he investigated the moduli of abelian surfaces on which there exist extra curves (i.e., their Néron-Severi group $\neq \mathbf{Z}$). Also Picard had come across these groups.

Hilbert himself worked on the problem around 1893. He proposed his student Blumenthal to work on the function theory of the quotients $SL(2, O_K)\backslash H^n$ and gave him his unpublished notes, which according to Blumenthal mainly dealt with theta functions for the Hilbert modular group. Blumenthal established among other things the existence of a fundamental domain [1]. Also Hecke wrote a thesis on the subject in which he constructed unramified abelian extensions of a bi-quadratic field using Hilbert modular functions [4]. But despite the efforts of these mathematicians the subject did not flourish. From a modern viewpoint it is easy to see why.

The quotients $SL(2, O_K)\backslash H^n$ are normal complex spaces with finitely many (innocent) quotient singularities. They admit a natural compactification obtained by adding finitely many points, called the cusps, one for each ideal class of $K$. The compactified quotients are normal complex spaces, and even algebraic varieties which can be imbedded in projective space using modular forms of high weight. These varieties are called Hilbert modular varieties. The cohomology of these varieties is described to a very large extent by Hilbert modular forms. As it turns out, the cusps are highly singular for $n > 1$. Although many concepts from the theory of the classical modular group like modular form, cusps, cusp form, Eisenstein series generalize immediately, without a mature theory of analytic functions of several complex variables and a well-developed theory of intrinsic algebraic geometry these varieties are inaccessible to further study. Therefore, a fruitful theory of these Hilbert modular varieties had to wait for new developments in algebraic geometry. To illustrate this point, we could refer to Hecke who asserts in his thesis [4] to have found nonconstant holomorphic functions on the quotients $SL(2, O_K)\backslash H^2$ (for $K$ real quadratic), thus contradicting facts from complex algebraic geometry that are now commonplace.

For quite some time there was not much activity in the field. Then, at the end of the sixties, there was a revival of interest in the theory of modular functions, first in one variable, but soon also for automorphic forms on more general arithmetic groups. Serre asked Hirzebruch in a letter whether he knew how to resolve the cusp singularities of the Hilbert modular varieties for real quadratic fields. Hirzebruch's immediate answer consisted of a long letter (January 18, 1971) in which he sketched the resolution he had just obtained a few days earlier. This coincidence marked the revival of the Hilbert modular group. The resolution of singularities (first in the two-dimensional case [6], later in the general case) gave a strong impetus. It became possible to compute invariants, and (for the quadratic case) to determine the type of surfaces that occurred. For example, up to 1970 the rationality of only a few Hilbert modular surfaces $SL(2, O_K) \setminus H^2$ was known, but soon afterwards there was a complete classification of all of them (almost all are of general type). The activity fitted well with the developments in the theory of automorphic forms; for example, the lifting of modular forms developed by Shintani, Langlands, and others (after pioneering work of Doi and Naganuma) could be given a geometric interpretation in work of Hirzebruch and Zagier [7].

Hilbert modular varieties are the moduli spaces of abelian varieties whose endomorphism ring contains $O_K$. This aspect played almost no role in the developments just mentioned, but it does in the extension given a few years later by Rapoport of the resolution of singularities to the arithmetic case [8]. So the compactified Hilbert modular varieties exist over rings of integers in a number field.

What became of Hilbert's expectations? Hilbert expected that research on the Hilbert modular varieties would stimulate function theory and algebraic geometry, but it worked the other way around. Kronecker's Jugendtraum found a realization in class field theory and the theory of canonical models of Shimura, though in a way quite different from what Hilbert expected. In my opinion the importance of Hilbert modular varieties nowadays comes from several facts. First, since the Hilbert modular group is the easiest generalization of the classical modular group, it may serve as a guiding example of more general and much more difficult arithmetic groups, like the symplectic group $\mathrm{Sp}(2g)$. Second, the Hilbert modular varieties are moduli spaces and this means that their points, subvarieties, etc. have a modular interpretation and—as noted above—that they are defined over a number field. This makes the Hilbert modular varieties much more accessible than arbitrary algebraic varieties. A good illustration of this is the fact that it is possible to express their $L$-functions in terms of modular forms. This fact together with the availability of lots of explicit algebraic curves on Hilbert modular surfaces has led to a proof of the Tate conjectures (describing the algebraic cycles as the Galois invariant ones and relating their existence to poles of the

$L$-function) for Hilbert modular surfaces by Harder, Langlands, and Rapoport [3]. Hilbert modular varieties may thus serve as a testing site for the many conjectures of (arithmetic) algebraic geometry. Finally, the Hilbert modular varieties provide beautiful examples of algebraic varieties as is amply illustrated by the case of Hilbert modular surfaces (cf. [2]).

As the title of the book under review indicates, it deals with Hilbert modular forms. It contains three chapters: Hilbert Modular Forms, Dimension Formulae, and The Cohomology of the Hilbert Modular Group. It is the purpose of the book to make the reader understand the singular cohomology groups of the Hilbert modular varieties $SL(2, O_K) \setminus H^n$. By this the author means determining their dimensions, the Hodge filtration, and the weight filtration. There are two methods to determine the dimensions: by the Selberg Trace Formula and by the Theorem of Riemann-Roch. The book uses the first one and follows a paper of Shimizu from 1963. The cohomology is expressed in three parts: the part coming from the cusp forms, the part coming from the Chern forms, and the one coming from the Eisenstein series as described by Harder. The Hodge numbers of Hilbert modular varieties are computed in a section written by C. Ziegler. The book stresses the notions which generalize directly from the classical modular group and its scope is limited: it does not touch upon the deeper properties of modular forms and cohomology. For example, this book on modular forms does not contain a single reference to the notion of Hecke operator. The book seems intended for graduate students and is indeed very clearly written. The author succeeds very well in explaining how the cohomology of Hilbert modular varieties is computed. However,it is unfortunate that the author never explains why the topic he deals with deserves our attention.

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*Transcendental numbers*, by Andrei B. Shidlovskii. Studies in mathematics, vol. 12, Walter de Gruyter, Berlin, New York, 1989, xix + 466 pp., $89.50. ISBN 3-11-011568-9

## 1

The title may suggest that the book deals with the general theory of transcendental numbers. A complex number $\alpha$ is said to be *algebraic* if it is a root of a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with rational coefficients and $f(x) \not\equiv 0$. If $\alpha$ is not algebraic, it is called *transcendental*. In 1874, Cantor showed that the set of all algebraic numbers is countable so that transcendental numbers exist. The first rigorous proof of the existence of transcendental numbers was given thirty years earlier by Liouville. We say that $\alpha$ is of *degree* $n$, if the smallest degree of polynomials $f$ as described above equals $n$. Liouville proved the existence of a positive constant $c(\alpha)$ such that every pair of rational integers $p$, $q$ with $q > 0$ and $p/q \neq \alpha$ satisfies

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^n} \quad (n \text{ is degree of } \alpha). \tag{1}$$

It is an easy consequence that numbers with very good rational approximations, such as $\sum_{n=1}^{\infty} 2^{-n!}$, are transcendental. After successive improvements of the exponent $n$ due to Thue (1909), Siegel (1921) and Dyson, Gelfond (1947/1948), Roth (1955)