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1

The title may suggest that the book deals with the general theory of transcendental numbers. A complex number α is said to be *algebraic* if it is a root of a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with rational coefficients and $f(x) \not\equiv 0$. If α is not algebraic, it is called *transcendental*. In 1874, Cantor showed that the set of all algebraic numbers is countable so that transcendental numbers exist. The first rigorous proof of the existence of transcendental numbers was given thirty years earlier by Liouville. We say that α is of *degree* n, if the smallest degree of polynomials f as described above equals n. Liouville proved the existence of a positive constant $c(\alpha)$ such that every pair of rational integers p, q with $q > 0$ and $p/q \neq \alpha$ satisfies

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^n} \quad (n \text{ is degree of } \alpha). \tag{1}$$

It is an easy consequence that numbers with very good rational approximations, such as $\sum_{n=1}^{\infty} 2^{-n!}$, are transcendental. After successive improvements of the exponent n due to Thue (1909), Siegel (1921) and Dyson, Gelfond (1947/1948), Roth (1955) proved, for any $\varepsilon > 0$, the existence of a constant $c(\alpha, \varepsilon) > 0$ such that for every pair p, q as described above

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha, \varepsilon)}{q^{2+\varepsilon}}. \tag{2}$$

The number 2 in the exponent cannot be further improved. It is a direct consequence that $\alpha = \sum_{n=1}^{\infty} 2^{-3^n}$ and similar numbers are transcendental. Roth's result (2) was generalized by Schmidt in 1971 to the case of approximation of an algebraic number by algebraic numbers of lower degree. This deep and useful branch of the theory of Diophantine approximations, which is still in fast development, is only touched upon in Chapter 1 of the book under review. A reader interested in this area is referred to lecture notes of W. M. Schmidt [9].

Proving the transcendence of specific numbers, such as e, π, e^{π}, $\log 2$, $\log 3/\log 2$, $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$ and the constant of Euler, γ, is a completely different problem. (The transcendence of the latter two numbers is still undecided.) In 1744, Euler stated without proof that a number of the form $\log b/\log a$, where a and b are positive rational numbers with b not equal to a rational power of a, must be a transcendental number. In 1837, Wantzel showed that the line segments which can be constructed by ruler and compass have lengths which can be expressed in terms of numbers obtained by successively solving a series of quadratic equations and are therefore algebraic. Thus the transcendence of π implies the impossibility of "squaring the circle" by ruler and compass, thereby solving a problem of antiquity in the negative. Hermite proved the transcendence of e in 1873, Lindemann the transcendence of π in 1882 by developing Hermite's method further. These results are contained in the theorem of Lindemann-Weierstrass: if the algebraic numbers $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbf{Q}$, then the numbers $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbf{Q}$. (Complex numbers $c_1, \dots, c_n$ are said to be *algebraically independent* over a field K if $P(c_1, \dots, c_n) \neq 0$ for any polynomial $P(z_1, \dots, z_n)$ which is not identically zero and has coefficients from K.)

In 1929, Gelfond showed that e^{π} is transcendental by a new analytic method involving interpolation techniques. By pursuing Gelfond's ideas, Gelfond and Schneider derived in 1934, independently of each other, a proof of Euler's assertion on $\log b/\log a$.

Their results also imply the solution of the seventh problem posed by Hilbert in his famous 1900 address to the International Congress of Mathematicians, namely the transcendence of α^β where α is an algebraic number not 0 or 1 and β is an algebraic irrational number. In 1966/1967, Baker succeeded in obtaining a far-reaching generalization of the Gelfond-Schneider theorem. He proved the transcendence of numbers of the form $\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$ and $\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$ under certain natural conditions on the algebraic numbers $\alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n$. This was the starting point of a period of more than twenty years of research in which the method has proved to be of great importance. This research is hardly mentioned in the book. Introductions to the part of transcendental number theory dealing with the Gelfond-Schneider method and its extensions have been given by Waldschmidt [13, 14], Baker [1], Masser [8], and Feldman [4].

The method of Hermite-Lindemann is based on two properties of the exponential function, namely, e^z satisfies the functional equation $f(x)f(y) = f(x+y)$ and e^z satisfies the differential equation $y' = y$. In 1929 Siegel developed a method for proving transcendence of the values at algebraic points of solutions of a linear differential equation with coefficients in the field of rational functions. Siegel did not require that the functions satisfy a functional equation. The solutions to which Siegel's method applies have to satisfy certain arithmetic conditions and are called E-functions. In 1949 Siegel [12] presented his method in the form of a general theorem on E-functions. This theorem reduces the proof of algebraic independence of certain numbers to the verification of a certain "normality condition." Siegel was able to verify this normality condition only in the case of E-functions satisfying a first or second order linear differential equation. In 1954, Shidlovskii was able to reduce the proof of the arithmetic property to the verification of a certain irreducibility condition for the functions. One year later, he showed that algebraic independence of certain functions over the field of rational functions is a necessary and sufficient condition for the algebraic independence of the values of these functions at algebraic points. This made it possible to prove transcendence and algebraic independence of the values at algebraic points of many concrete E-functions which are solutions of linear differential equations of arbitrary order. Shidlovskii's book contains an exclusive exposition of the fundamental results on the arithmetic properties of the values of E-functions. The most complete account on the subject in English up to now had been given by Mahler [7].

Thus, the book under review does not deal with the general theory of transcendental numbers, covering topics as the Thue-Siegel-Roth-Schmidt method, Gelfond-Baker method, Siegel-Shidlovskii method, and various other topics which belong to the field such as Mahler's classification, metrical theory, elliptic functions, and abelian varieties. For such books, see Siegel [12], Schneider [10], Gelfond [5], Lang [6], Baker [1], and Feldman [3].

2

A proper title of the book would have been "The Siegel-Shidlovskii method," or "E-functions." Siegel's contribution is contained in two fundamental publications, his famous 1929 paper [11] and his book [12] published in 1949. Shidlovskii's contribution consists of a series of about forty papers published since 1954. Further contributions have been made by the Soviet mathematicians Belogrivov, Chirskii, Galochkin, Gorelov, Kazakov, Makarov, Nesterenko, Oleinikov, Salikhov, Shmelev and further Lang, Mahler, Osgood, Wallisser and Xu. Since many publications are only available in Russian, the appearance of Shidlovskii's book in English makes the literature on the subject more readily available.

Already Legendre had considered the functions

$$(3)\qquad f_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\alpha(\alpha+1)\cdots(\alpha+n-1)} \qquad (\alpha \notin \mathbf{Z}_{<0})$$

which are solutions of the differential equation $xy''+\alpha y' = y$, and had proved that the numbers $f_\alpha(x)/f'_\alpha(x)$ are irrational if α and x are rational, $x \neq 0$. In 1910 Stridsberg showed that each of the numbers $f_\alpha(x)$ and $f'_\alpha(x)$ is irrational. A basic result which Siegel obtained in 1929 relates to the functions

$$(4)\quad K_\lambda(z) = f_{\lambda+1}\left(\frac{-z^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\lambda+1)\cdots(\lambda+n)}\left(\frac{z}{2}\right)^{2n} \qquad (\lambda \notin \mathbf{Z}_{<0})$$

which satisfy the differential equation $y'' + \frac{2\lambda+1}{z}y' + y = 0$. The function $K_\lambda(z)$ differs only by the factor $\Gamma(\lambda+1)^{-1}(z/2)^{2\lambda}$ from the Bessel function $J_\lambda(z)$, and $K_0(z) = J_0(z)$. Siegel proved the following analog of the theorem of Lindemann-Weierstrass: The

2mn numbers $K_{\lambda_h}(\xi_j)$, $K'_{\lambda_h}(\xi_j)(h=1,\dots,m;\, j=1,\dots,n)$ *are algebraically independent, provided that* $\xi_1,\dots,\xi_n$ *are algebraic numbers whose squares are distinct and nonzero and that the rational numbers* $\lambda_1,\dots,\lambda_m$ *satisfy the conditions* $\lambda_h \notin \mathbf{Z}_{<0}$, $\lambda_h+\frac{1}{2} \notin \mathbf{Z}$ *and* $(\lambda_{h_1} \pm \lambda_{h_2}) \notin \mathbf{Z}$ *for* $h_1 \neq h_2$. In addition, he obtained a positive lower bound for the modulus of a polynomial with integral coefficients in $J_0(\xi)$ and $J'_0(\xi)$ for nonzero algebraic ξ. A result of the latter type is called an *algebraic independence measure.*

Siegel's book does not contain any new concrete results, but Siegel presented his method in the form of a general theorem on E-functions. An analytic function $f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ is called an E-function if (1) the coefficients c_n are taken from an algebraic number field K, (2) for any $\varepsilon > 0$ the conjugates of c_n have a growth order bounded by $n^{\varepsilon n}$, (3) the coefficients c_n have denominators of growth order bounded by $n^{\varepsilon n}$. Simple examples of E-functions are e^z, $\sin z$, $J_0(z)$, $f_\alpha(z)$ and $K_\alpha(z)$ for $\alpha \in \mathbf{Q}\backslash\mathbf{Z}_{<0}$. The E-functions form a ring which is closed under the operations of differentiation, integration from 0 to z, and the change of variables from z to λ, where λ is an algebraic number. Siegel's general theorem involves a certain analytic normality condition for sets of products of powers of the functions under consideration which in concrete examples was hard to verify. In 1955 Shidlovskii reduced the algebraic independence of the values of E-functions to the algebraic independence of the functions themselves over the field of rational functions. This theorem made it possible to prove transcendence and algebraic independence of the values at algebraic points of many concrete E-functions which are solutions of linear differential equations of arbitrary order. One of Shidlovskii's fundamental theorems can be stated as follows.

Let

$$Y'_h = Q_{h0} + \sum_{k=1}^{m} Q_{hk} Y_k \qquad (h=1,\dots,m) \tag{5}$$

be a system of linear differential equations with coefficients in the field of rational functions $\mathbf{C}(z)$. *Suppose that the E-functions* $f_1(z),\dots,f_m(z)$ *form a solution of system* (5). *Let* $\alpha \neq 0$ *be any algebraic number different from the poles of the functions* Q_{h0} *and* Q_{hk}. *Then the largest number of functions* $f_1(z),\dots,f_m(z)$ *that are algebraically independent over* $\mathbf{C}(z)$ *is equal to the largest number of function values* $f_1(\alpha),\dots,f_m(\alpha)$ *that are algebraically independent over* $\mathbf{Q}$, *in other words, the transcendence degree of*

$f_1(z), \ldots, f_m(z)$ over $\mathbf{C}(z)$ is equal to the transcendence degree of $f_1(\alpha), \ldots, f_m(\alpha)$ over $\mathbf{Q}$.

The book under review is devoted to Shidlovskii's fundamental theorems and their applications. Chapter 1 deals with approximation by algebraic numbers. Chapter 2 contains a proof of the Theorem of Lindemann-Weierstrass. In Chapters 3 and 4 three versions of Shidlovskii's Fundamental Theorem are given. Chapters 5–10 provide applications to linear differential equations of first, second, prime and arbitrary order. Chapters 11–13 deal with transcendence and algebraic independence measures. The treatment is thorough and complete, but sometimes a bit boring for nonexperts or nonfanatics. The scope is rather narrow. There is even hardly attention for Siegel's G-functions which have been extensively studied in the past twenty years. (A *G-function* is an analytic function $\sum_{n=0}^{\infty} c_n z^n$ satisfying the same three conditions as E-functions; only the factor $(n!)^{-1}$ is omitted and some authors replace the bound $n^{\varepsilon n}$ by e^{cn} where c is an arbitrarily large constant.) The results on G-functions are summarized in the "Concluding remarks." Another example of the narrow view can be found on p. 435: "We note that the size of the constant τ in Lemma 8 has been lowered in papers by Brownawell [8:1], Bertrand and Beukers [4:1], Nesterenko [52:11], and others. In this book we used Nesterenko's first result, because it is more accessible to the Soviet reader." The Soviet reader may be interested in at least knowing the value of τ in the various improvements. The given argument has an opposite effect for the users of the English translation of Shidlovskii's book. Of course, the important point in publishing this English translation is that an important method in transcendental number theory has been made better accessible. There is a list of about 250 references.

3

The original Russian version of Shidlovskii's book was published in 1987. In 1988 Beukers, Brownawell, and Heckman [2] showed that the essential part of Siegel's normality condition can be expressed very naturally in terms of the differential Galois group corresponding to the system (5). Their observation made it possible to verify Siegel's normality condition for large classes of generalized hypergeometric equations. This and other recent developments are mentioned in a Foreword by W. D. Brownawell (3 pp.) and in "Supplementary remarks on recent work for the English edition" (8 pp.) and 25 additional references.

Siegel's general theorem can be stated as follows [12, pp. 52–53]. *Let*

$$(6) \qquad Y_h' = \sum_{k=1}^{m} Q_{hk} Y_k \qquad (h = 1, \dots, m)$$

be a system of linear differential equations with coefficients in $\mathbf{C}(z)$. *Suppose that the E-functions* $f_1(z), \dots, f_m(z)$ *form a solution of system* (6) *and that the* $\binom{m+\nu}{m}$ *power products*

$$f_1^{\nu_1} \cdots f_m^{\nu_m} \ (\nu_1 + \cdots + \nu_m \le \nu)$$

form a normal system for all $\nu = 1, 2, \dots$. *Let* $\alpha \ne 0$ *be any algebraic number different from the poles of the functions* Q_{hk}. *Then the function values* $f_1(\alpha), \dots, f_m(\alpha)$ *are algebraically independent over* $\mathbf{Q}$. The rather complicated definition of normal system can be
found on pp. 43–44 of Siegel's book and on pp. 446–447 of the book under review. Beukers, Brownawell, and Heckman [2] showed that Siegel's normality condition can be expressed very naturally in terms of differential Galois groups. Siegel's normality condition for an irreducible differential equation (6) turns out to be equivalent with the requirement that the differential Galois group of (6) contains either $\mathrm{SL}(m)$ or $\mathrm{Sp}(m)$. Another characterization is that there exists a solution $f_1, \dots, f_m$ such that f_i/f_j is transcendental for some i and j and the symmetric square of (6) is irreducible. By these characterizations they were able to prove a new and very general algebraic independence result of which the following generalization of the theorem of Lindemann-Weierstrass is a special case: A parameter set S of real numbers $\{\mu_1, \dots, \mu_p; \nu_1, \dots, \nu_q\}$, where $q > p \ge 0$ and $\nu_q = 1$, is called *admissible* if it satisfies at least one of the following conditions:

(A) $\nu_j - \mu_i \notin \mathbf{Z}$ for $1 \le i \le p$, $1 \le j \le q$ and all sums $\nu_i + \nu_j$ $(1 \le i \le j \le q)$ are distinct mod $\mathbf{Z}$.

(B) $p = 0$; q is odd or $q = 2$; and the set $\{\nu_1, \dots, \nu_q\}$ is modulo $\mathbf{Z}$ not a union of arithmetic sequences $\{\nu, \nu + 1/d, \dots, \nu + (d-1)/d\}$ for a fixed length d where $d|q$, $d > 1$.

For a given parameter set S we consider the function

$$f_S(z) = \sum_{n=0}^{\infty} \frac{(\mu_1)_n \cdots (\mu_p)_n}{(\nu_1)_n \cdots (\nu_{q-1})_n} \cdot \frac{(-z)^{(q-p)n}}{n!}$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. (Hence f_s is a generalized hypergeometric function of type ${}_pF_{q-1}$ with z replaced by $(-z)^{q-p}$.) *Let S be an admissible parameter set with rational μ's and ν's. Let b be a nonzero algebraic number. Let $\alpha_1, \dots, \alpha_s$ be algebraic numbers which are linearly independent over* $\mathbf{Q}$. *Then the numbers*

$$e^{\alpha_1}, \dots, e^{\alpha_s}, f_S(b), f_S'(b), \dots, f_S^{(q-1)}(b)$$

are algebraically independent over $\mathbf{Q}$.

A more general statement deals with r parameters sets $S_1, \dots, S_r$. Using recent work of N. M. Katz and O. Gabber it is possible to weaken the conditions (A) and (B) considerably. See [15] or Chapter 3 on the generalized hypergeometric function of the recently published book [16].

It is not yet clear where this new development will bring us, but there are many systems of equations to which Shidlovskii's method can be applied successfully, but which are not Siegel normal. These systems have a nonreductive differential Galois group. Not much systematic work has been done on this nonreductive case yet, but it seems that Differential Galois Theory will also play an important role there. Using some simple techniques from this theory D. Bertrand proved the following remarkable theorem. *Let*

$$Y^{(n)} + p_1 Y^{(n-1)} + \cdots + p_n Y = q$$

be an inhomogeneous differential equation with $p_1, \dots, p_n$, $q \in$ $\mathbf{C}(z)$. *Suppose that it has no solution in* $\mathbf{C}(z)$ *and that the homogenized equation is irreducible. Then for any solution $f(z)$ the functions $f, f', \dots, f^{(n-1)}$ are algebraically independent over* $\mathbf{C}(z)$. These new developments will increase the importance of the Siegel-Shidlovskii theory. As Brownawell puts it in his Foreword, "Shidlovskii's book will be recognized as a classic of the Siegel-Shidlovskii theory. Its accessible introduction to and authoritative survey of the field provides a solid foundation for continuing progress."

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