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Twistor theory for Riemannian symmetric spaces by Francis E. Burstall and John H. Rawnsley. Springer-Verlag, Berlin and New York, 1990, 112 pp., $14.70. ISBN 3-540-52602-1

Twistor theory originated in the work of Roger Penrose in the 1960s and has developed in a number of directions. Its original context was mathematical physics, and for this aspect there are several recent surveys [9, 5, 11]. The monograph under review has very little to do with twistor theory as applied to physics and as discussed in the above references. In particular the authors state in the first sentence of their introduction: "The subject of this monograph is the interaction between real and complex homogeneous geometry and its application to the study of minimal surfaces (or harmonic maps)." The word twistor theory refers to a particular use of a twistor space associated to a Riemannian manifold in order to generate minimal surfaces or harmonic maps from holomorphic maps into the twistor space. We will say more about this below.

Weierstrass noticed in the nineteenth century that one could locally define minimal surfaces by means of holomorphic curves, i.e., any triple of holomorphic functions

$$(f_1, f_2, f_3): \mathbf{C} \to \mathbf{C}^3$$

which satisfies

$$f_1^2 + f_2^2 + f_3^2 = 0,$$

determines locally a minimal surface in $\mathbf{R}^3$. This is given by the formula

$$F(z) = \mathrm{Re}\left(\int_{z_0}^{z} (f_1, f_2, f_3)\, dz\right).$$

Moreover, all minimal surfaces in $\mathbf{R}^3$ admit such a parameterization locally. This led to significant efforts in the twentieth century to represent minimal surfaces and more generally harmonic maps in terms of holomorphic objects. The monograph by Burstall and Rawnsley presents a sophisticated, elegant, and in-depth look at this question.

In the study of instantons on the 4-sphere the fibration $Z = \mathbf{P}_3(\mathbf{C}) \to S^4$ played a fundamental role in the existence theory and classification questions (see [11] for details and references). The space Z is called the *twistor space* of the 4-sphere S^4. This particular fibration (defined naturally by the canonical mapping $\mathbf{P}_3(\mathbf{C}) \to \mathbf{P}_1(\mathbf{H}) \cong S^4$, obtained by choosing a quaternionic structure on $\mathbf{C}^4$), relates naturally to all of the developments in twistor theory that relate to mathematical physics. The map has fibres which are one-dimensional complex projective spaces, and Bryant [1] made the observation that maps of the form

$$\phi: S^2 \to Z$$

which were holomorphic and transversal to the fibres and satisfying an additional holomorphic differential condition of first order, projected to minimal immersions of a 2-sphere in S^4. Thus the problem of constructing minimal immersions in S^4 was reduced to the construction of holomorphic objects in the associated twistor space. This theorem of Bryant (and others in similar contexts by other mathematicians) led to the search by the authors for a general family of "twistor spaces" $Z \to M$ for M a Riemannian manifold with the property that harmonic maps

$$\phi: N \to M$$

were realized as projections of "holomorphic" maps

$$\tilde{\phi}: N \to Z$$

satisfying additional geometric and infinitesimal conditions.

In all cases the twistor space is fibred over the target Riemannian manifold M, and the space Z and the fibres are almost complex manifolds, sometimes with more than one almost-complex structure of interest at the same time, and the notion of "holomorphic mapping" from the domain space N (almost always a Riemann surface with its usual complex structure) to the twistor space Z is in the sense of preserving the almost-complex structures.

The class of manifolds M for which this methodology is successfully carried out in this monograph are the inner Riemannian symmetric spaces. Here *inner* means that the involution required in the definition of a Riemannian symmetric space is an inner automorphism. This is equivalent to $M = G/K$ being inner if and only if $\operatorname{rank} G = \operatorname{rank} K$, in the standard representation of M as a quotient of a semi-simple Lie group G by a maximal compact subgroup K. These spaces are always even-dimensional and include the even-dimensional spheres and the Hermitian symmetric spaces of semisimple type.

The authors construct twistor spaces which are flag manifolds (generalizations of classical complex Grassmannian manifolds), which are defined in terms of semisimple Lie groups and their complexifications. Flag domains also play a role in this context. These are open subsets of a flag manifold which are open orbits of real forms of the complex transitive groups involved (e.g. the upper half-plane is the orbit of $\mathrm{SL}(2, \mathbf{C})$ action on $\mathbf{P}_1(\mathbf{C}) \cong \mathrm{SL}(2, \mathbf{C})/P$, where P is a parabolic subgroup).

The book divides into three parts:

Homogeneous geometry. This is a study of all of the homogeneous spaces and their properties used in the later parts. This includes very nice representations for the Levi-Cevita connection, using a generalization of the 1-form of Maurer-Cartan for the homogeneous spaces. This gives very useful representations for curvature, torsion, and other differential-geometric objects of interest which occur in the theory of harmonic maps.

Twistor theory. Here twistor theory means twistor spaces which fibre over Riemannian manifolds, and in which harmonic maps lift to holomorphic ones as described briefly above. The existence of such twistor spaces is delicate and uses the full strength of the root structure for complex semisimple Lie algebras which is summarized in a suitable form in a special chapter.

Harmonic maps. Here the machinery of the previous chapters is put to use. To find existence of suitable holomorphic objects in the twistor space, the authors make full use of the decomposition theorem of Grothendieck [6] for vector bundles on Riemann surfaces into the direct sum of line bundles of a specific form. A similar filtration of vector bundles due to Harder and Narasimhan [7] is also used to provide existence theorems. Previous work of Calabi [3], Eells-Wood [4], and Bryant [2] is subsumed in one general proof using these methods. The authors give a further application of these ideas for harmonic maps into simple Lie groups (belonging to a large class of such groups), generalizing work of Uhlenbeck [10] concerning harmonic maps of $S^2 \to \mathrm{U}(n)$.

The book appears to be very carefully written but will be tough going for those not conversant with standard Lie theory and structure theory as represented by Helgason's classical book [8]. Most of the technical work is done at the Lie algebra level and that is part of the virtue in that a difficult problem in nonlinear analysis admits algebraic solutions in this context.

The introduction gives a good outline of the material and provides a guide through the various technical chapters (Chapters 1–5) leading up to the culmination in the last three chapters where existence and classification of harmonic mappings is given in a wide range of circumstances. This work includes new results on stable harmonic 2-spheres which is joint work with Simon Salomon. The authors are to be commended for providing valuable insight into a beautiful and mysterious subject: how to construct harmonic maps from holomorphic ones.

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