

## $\Lambda$ -TREES AND THEIR APPLICATIONS

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To most mathematicians and computer scientists the word “tree” conjures up, in addition to the usual image, the image of a connected graph with no circuits. We shall deal with various aspects and generalizations of these mathematical trees. (As Peter Shalen has pointed out, there will be leaves and foliations in this discussion, but they do not belong to the trees!) In the last few years various types of trees have been the subject of much investigation. But this activity has not been exposed much to the wider mathematical community. To me the subject is very appealing for it mixes very naïve geometric considerations with the very sophisticated geometric and algebraic structures. In fact, part of the drama of the subject is guessing what type of techniques will be appropriate for a given investigation: Will it be direct and simple notions related to schematic drawings of trees or will it be notions from the deepest parts of algebraic group theory, ergodic theory, or commutative algebra which must be brought to bear? Part of the beauty of the subject is that the naïve tree considerations have an impact on these more sophisticated topics. In addition, trees form a bridge between these disparate subjects.

Before taking up the more exotic notions of trees, let us begin with the graph-theoretic notion of a tree. A graph has vertices and edges with each edge having two endpoints each of which is a vertex. A graph is a simplicial tree if it contains no loops. In §1 we shall discuss simplicial trees and their automorphism groups. This study is closely related to combinatorial group theory. The later sections of this article shall focus on generalizations of the notion of a simplicial tree. A simplicial tree is properly understood to be a  $\mathbf{Z}$ -tree. For each ordered abelian group  $\Lambda$ , there is an analogously defined object, called a  $\Lambda$ -tree. A very important special case is when  $\Lambda = \mathbf{R}$ . When  $\Lambda$  is not discrete, the automorphisms group of a  $\Lambda$ -tree is no longer combinatorial in nature. One finds mixing (i.e., ergodic) phenomena occurring, and the study of the automorphisms is much richer and less well understood. We shall outline this more general theory and draw parallels and contrasts with the simplicial case.

While the study of trees and their automorphism groups is appealing *per se*, interest in them has been mainly generated by considerations outside the subject. There are several basic properties of trees that account for these connections. As we go along we shall explain these notions and their connections in more detail but let me begin with an overview.

**1. One-dimensionality of trees.** Trees are clearly of dimension one. This basic property is reflected in relations of trees to both algebraic and geometric objects.

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(a) *Algebraic aspects.* The resurgence of the study of trees began with Serre's book [20] on  $SL_2$  and trees. In this book, Serre showed how there is naturally a tree associated to  $SL_2(K)$  where  $K$  is a field with a given discrete valuation. The group  $SL_2(K)$  acts on this tree. More generally, if  $G(K)$  is a semisimple algebraic group of real rank 1 (like  $SO(n, 1)$ ) and if  $K$  is a field with a valuation with formally real residue field, then there is a tree on which  $G(K)$  acts. The reason for this can be summarized by saying that the local Bruhat-Tits building [2] for a rank 1 group is a tree. The fact that the group is rank 1 is reflected in the fact that its local building is one-dimensional. (One does not need to understand to entire Bruhat-Tits machinery to appreciate this connection. In fact, one can view this case as the simplest, yet representative, case of the Bruhat-Tits theory.)

(b) *Codimension-1 dual objects.* Suppose that  $M$  is a manifold whose fundamental group acts on a tree. Since the tree is 1-dimensional there is a dual object in the manifold which is codimension-1. This object turns out to be a codimension-1 lamination with a transverse measure. These dynamic objects are closed related to isometry groups of trees. In fact, dynamic results for these laminations can be used to study group actions on trees. They also give rise to many interesting examples of such actions.

2. **Negative curvature of trees.** Suppose that we have a simply connected Riemannian manifold with distance function  $d$  of strictly negative curvature. In  $M$  the following geometric property holds: If  $A_1, A_2, B_1, B_2$  are points with  $d(A_1, A_2)$  and  $d(B_1, B_2)$  of reasonable size but  $d(A_1, B_1)$  extremely large, the geodesics  $\gamma_1$  and  $\gamma_2$  joining  $A_1$  to  $B_1$  and  $A_2$  to  $B_2$  are close together over most of their length. (See Figure 1.) The estimate on how close will depend on an upper bound for the curvature and will go to 0 as this bound goes to  $-\infty$ . In fact, Gromov [8] has defined a class of negatively curved metric spaces in terms of this 4-point property. From this point of view  $\mathbf{R}$ -trees are the most negative curved of all spaces, having curvature  $-\infty$ , since in a tree the geodesics  $\gamma_1$  and  $\gamma_2$  coincide over most of their lengths.

We state this fact in another way. Suppose that we have a sequence on strictly negatively curved, simply connected manifolds with curvature upper bounds going  $-\infty$ , then a subsequence of these manifolds converges in a geometric sense to an  $\mathbf{R}$ -tree. If the manifolds in question are the universal coverings of manifolds with a given fundamental group  $G$ , the actions of  $G$  of the manifolds in the subsequence converge to an action of  $G$  on the limit tree. The space of negatively curved manifolds with fundamental group  $G$  is completed by adding ideal points at infinity which are represented by actions of  $G$  on  $\mathbf{R}$ -trees. This

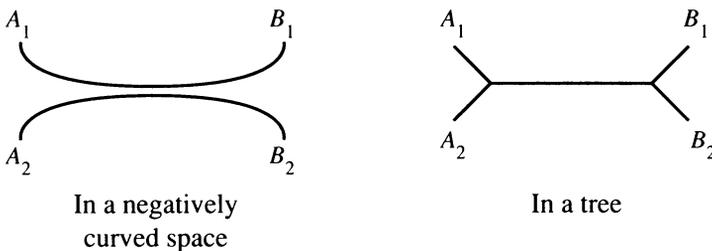


FIGURE 1

is of particular importance for the manifolds of constant negative curvature—the hyperbolic manifolds. This of course brings us full circle since the group of automorphisms of hyperbolic  $n$ -space is the real rank-1 group  $SO(n, 1)$ .

The paper is organized in the following manner. The first section is devoted to discussing the now classical case of simplicial trees and their relationship to combinatorial group theory and to  $SL_2$  over a field with a discrete valuation. Here we follow [20] closely. This material is used to motivate all other cases.

Section 2 begins with the definition of a  $\Lambda$ -tree for a general ordered abelian group  $\Lambda$ . We show how if  $K$  is a field with a nondiscrete valuation with value group  $\Lambda$ , then there is an action of  $SL_2(K)$  on a  $\Lambda$ -tree. More generally, for any semisimple algebraic group of real rank 1 there is a similar result. We explain the case of  $SO(n, 1)$  in some detail. The last part of the section discusses some of the basics of the actions of groups on  $\Lambda$  trees. We introduce the hyperbolic length of a single isometry and define and study some of the basic properties of the space of all projective classes of nontrivial actions. At the end of the section we discuss the concept of base change.

Section 3 discusses the application of this material to compactify the space of conjugacy classes of representations of a given finitely presented group into a rank-1 group such as  $SL_2$  or  $SO(n, 1)$ . We approach this both algebraically and geometrically. The main idea is that a sequence of representations of a fixed finitely generated group  $G$  into  $SO(n, 1)$  has a subsequence which converges modulo conjugation either to a representation into  $SO(n, 1)$  or to an action of  $G$  on an  $\mathbf{R}$ -tree. In the case when the representations are converging to an action on a tree, this result can be interpreted as saying that as one rescales hyperbolic space by factors going to zero, the actions of  $G$  on hyperbolic space coming from the sequence of representations of  $G$  into  $SO(n, 1)$  converge to an action of  $G$  on an  $\mathbf{R}$ -tree. This can then be used to compactify the space of conjugacy classes of representations of  $G$  in  $SO(n, 1)$  with the ideal points being represented as actions of  $G$  on  $\mathbf{R}$ -trees.

Section 4 we consider the relationship between  $\mathbf{R}$ -trees and codimension-1 measured laminations. We define a codimension-1 measured lamination in a manifold. We show how ‘most’ measured laminations in a manifold  $M$  give rise to actions of  $\pi_1(M)$  on  $\mathbf{R}$ -trees. We use these to give examples of actions of groups on  $\mathbf{R}$ -trees, for example, actions of surface groups. We also show how any action of  $\pi_1(M)$  on an  $\mathbf{R}$ -tree can be dominated by a measured lamination.

In the last section we give applications of the results from the previous sections to study the space of hyperbolic structures (manifolds of constant negative curvature) of dimension  $n$  with a given fundamental group  $G$ . We have a compactification of this space where the ideal points at infinity are certain types of actions of  $G$  on  $\mathbf{R}$ -trees. We derive some consequences—conditions under which the space of hyperbolic structures is compact. The theme running through this section is that to make the best use of the results of the previous sections one needs to understand the combinatorial group-theoretic information that can be derived from an action of a group on an  $\mathbf{R}$ -tree. For simplicial actions this is completely understood, as we indicate in §1. For more general actions there are some partial results (coming mainly by using measured laminations), but no general picture. We end the article with some representative questions about groups acting on  $\mathbf{R}$ -trees and give the current state of knowledge on these questions.

For other introductions to the subject on group actions on  $\Lambda$ -trees see [21, 12, 3, or 1].

1. SIMPLICIAL TREES, COMBINATORIAL GROUP THEORY, AND  $SL_2$

In this discussion of the ‘classical case’ of simplicial trees, we are following closely Serre’s treatment in *Trees* [20]. By an abstract 1-complex we mean a set  $V$ , the set of vertices, and a set  $E$ , the set of oriented edges. The set  $E$  has a free involution  $\tau: e \mapsto \bar{e}$ , called *reversing the orientation*. There is also a map

$$\partial: E \rightarrow V \times V, \quad \partial e = (i(e), t(e)),$$

which associates to each oriented edge its initial and terminal vertex satisfying  $i(\bar{e}) = t(e)$ . Closely related to an abstract 1-complex is its *geometric realization*, which is a topological space. It is obtained by forming the disjoint union

$$V \amalg \coprod_{e \in E} I_e,$$

where each  $I_e$  is a copy of the closed unit interval, and taking the quotient space under the following relations:

- (i)  $0 \in I_e$  is identified with  $i(e) \in V$ .
- (ii)  $1 \in I_e$  is identified with  $t(e) \in V$ .
- (iii)  $s \in I_e$  is identified with  $1 - s$  in  $I_{\bar{e}}$ .

The *order* of a vertex  $v$  is the number of oriented edges  $e$  for which  $i(e) = v$ .

A 1-complex is finite if it has finitely many edges and vertices. In this case its geometric realization is a compact space. Figure 2 gives a typical example of a finite 1-complex.

A nonempty, connected 1-complex is a *graph*; a simply connected graph is a *simplicial tree*. A graph is a simplicial tree if and only if it has no loops (topological embeddings of  $S^1$ ). Figure 3 gives the unique (up to isomorphism) trivalent simplicial tree. One aspect of the negative curvature of trees is reflected in the fact that the number of vertices of distance  $\leq n$  from a given vertex is  $3(2^n - 1) + 1$ , a number which grows exponentially with  $n$ .

The automorphism group of a simplicial tree  $T$ ,  $\text{Aut}(T)$ , is the group of all self-homeomorphisms of  $T$  which send vertices to vertices, edges to edges, and which are linear on each edge. This is exactly the automorphism group of the abstract complex  $(V, E, \partial, \tau)$ . An action of a group  $G$  on  $T$  is a homomorphism from  $G$  to  $\text{Aut}(T)$ . The action is said to be *without inversions* if for all  $g \in G$  and all  $e \in E$  we have  $e \cdot g \neq \bar{e}$ . To restrict to actions without

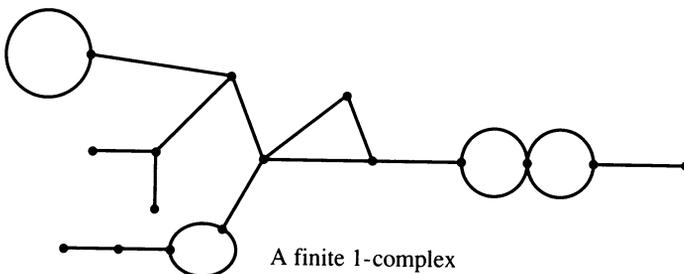


FIGURE 2

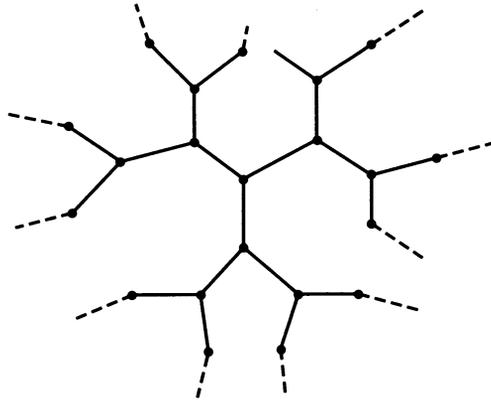


FIGURE 3

inversions is not a serious limitation since any action of  $G$  on  $T$  becomes an action without inversions on the first barycentric subdivision of  $T$  (obtained by splitting each edge of  $T$  into its two halves and adding a new vertex at the center of each edge).

Given an action of a group  $G$  on a simplicial tree  $T$  there is the quotient graph  $\Gamma$ . Its vertices are  $V/G$ ; its oriented edges are  $E/G$ . The geometric realization of this complex is naturally identified with the quotient space  $T/G$ .

**Graphs of groups.** We are now ready to relate the theory of group actions on simplicial trees to combinatorial group theory. The key to understanding the nature of an action of a group on a simplicial tree is the notion of a graph of groups. A graph of groups is the following:

- (i) A graph  $\Gamma$ ,
- (ii) for each oriented edge or vertex  $a$  of  $\Gamma$  a group  $G_a$  such that if  $e$  is an oriented edge then  $G_{\bar{e}} = G_e$ , and finally,
- (iii) if  $v = i(e)$  then there is given an injective homomorphism  $G_e \hookrightarrow G_v$ .

This graph of groups is said to be *over*  $\Gamma$ .

There is the fundamental group of a graph of groups. A topological construction of the fundamental group goes as follows. For each edge or vertex  $a$  of  $\Gamma$  choose a space  $X_a$  whose fundamental group is  $G_a$ . We can do this so that if  $v$  is a vertex of  $e$ , then there is an embedding  $X_e \hookrightarrow X_v$  realizing the inclusion of groups. We form the topological space  $X(\Gamma)$  by beginning with the disjoint union

$$\coprod_{v \in V} X_v \coprod \coprod_{e \in E} X_e \times I,$$

and (a) identifying  $X_e \times I$  with  $X_{\bar{e}} \times I$  via  $(x, t) \equiv (x, 1 - t)$  and (b) gluing  $X_e \times \{0\}$  to  $X_{i(e)}$  via the given inclusion. The resulting topological space has fundamental group which is the fundamental group of the graph of groups. A purely combinatorial construction of this fundamental group is given in [20, pp. 41–42].

**Example 1.** Let  $\Gamma$  be a single point. Then a graph over  $\Gamma$  is simply a group. Its fundamental group is the group itself.

**Example 2.** Let  $\Gamma$  be a graph with two vertices and a single edge connecting them. Then a graph of groups over  $\Gamma$  is the same thing as an embedding of the edge group into two vertex groups. Its fundamental group is the free product with amalgamation of the vertex groups over the edge group.

**Example 3.** Let  $\Gamma$  be a graph with one vertex and a single edge, forming a loop. Then graph of groups over  $\Gamma$  is a group  $G_v$  and two embeddings  $\varphi_0$  and  $\varphi_1$  of another group  $G_e$  into  $G_v$ . The fundamental group of this graph of groups is the corresponding HNN-extension. A presentation of this extension is

$$\langle G_v, s | s^{-1}\varphi_1(g)s = \varphi_0(g) \text{ for all } g \in G_e \rangle.$$

In general, the fundamental group of a graph of groups over a finite graph can be described inductively by a finite sequence of operations as in Examples 2 and 3. The fundamental group of an infinite graph of groups is the inductive limit of the fundamental groups of the finite subgraphs of groups.

Thus, it is clear that the operation of taking the fundamental group of a graph of groups generalizes two basic operations of combinatorial group theory—free product with amalgamation and HNN-extension.

There is a natural action of the fundamental group  $G$  of a graph of groups on a simplicial tree. To construct this action, let  $X$  be the topological space, as described above, whose fundamental group is the fundamental group of the graph of groups. There is a closed subset  $Y = \coprod_e X_e \times \{1/2\} \subset X$  which has a collar neighborhood in  $X$ . Let  $\tilde{X}$  be the universal covering of  $X$ , and let  $\tilde{Y} \subset \tilde{X}$  be the preimage of  $Y$ . We define the dual tree  $T$  to  $\tilde{Y} \subset \tilde{X}$ . Its vertices are the components of  $\tilde{X} - \tilde{Y}$ . Its unoriented edges are the components of  $\tilde{Y}$ . The vertices of an edge given by a component  $\tilde{Y}_0$  of  $\tilde{Y}$  are the two components of  $\tilde{X} - \tilde{Y}$  which have  $\tilde{Y}_0$  in their closure. The fact the  $\tilde{X}$  is simply connected implies that  $T$  is contractible and hence is a simplicial tree.

There is a natural action of  $G$  on  $\tilde{X}$ . This action leaves  $\tilde{Y}$  invariant and hence defines an action of  $G$  on  $T$ . It turns out that, up to isomorphism, this action of  $G$  on  $T$  is independent of all the choices involved in its construction. It is called *the universal action associated with the graph of groups decomposition of  $G$* . The quotient  $T/G$  is naturally identified with the original graph. Notice that the stabilizer of a vertex or edge  $a$  of  $T$  is identified, up to conjugation, with the group in the graph of groups indexed by the image of  $a$  in  $T/G$ .

Here, we see for the first time the duality relationship between trees and codimension-1 subsets.

**Structure theorem for groups acting on simplicial trees.** The main result in the theory of groups actions on simplicial trees is a converse to this construction for a graph of groups decomposition of  $G$ . It says:

**Theorem 1.** *Let  $G \times T \rightarrow T$  be an action without inversions of a group on a simplicial tree. Then there is a graph of groups over the graph  $T/G$  and an isomorphism from the fundamental group of this graph of groups to  $G$  in such a way that the action of  $G$  on  $T$  is identified up to isomorphism with the universal action associated with the graph of groups.*

**Corollary 2.** *Let  $H$  act on a simplicial tree  $T$ . Suppose that no point of  $T$  is fixed by all  $h \in H$ . The  $H$  has a HNN-decomposition or has a nontrivial decomposition as a free product with amalgamation.*

*Proof.* At the expense of subdividing  $T$  we can suppose that  $H$  acts without inversions. Consider the quotient graph  $T/H$ . We have a graph of groups over  $T/H$  whose fundamental group is identified with  $H$ . There is a minimal subgraph  $\Gamma \subset T/H$  such that the fundamental group of the restricted graph of groups over  $\Gamma$  includes isomorphically into  $H$ . This graph cannot be a single vertex since the action of  $H$  on  $T$  does not fix any point. If  $\Gamma$  has a separating edge  $e$ , there is a nontrivial free product with amalgamation decomposition for  $H$  as  $H_a$  and  $H_b$  amalgamated along  $G_e$ , where  $H_a$  and  $H_b$  are the fundamental groups of the graphs of groups over the two components of  $\Gamma - e$ . If  $\Gamma$  has a nonseparating edge  $e'$ , then there is an HNN-decomposition for  $H$  given by the two embeddings of  $G_e$  into the fundamental group of the graph of groups over  $\Gamma - e'$ .  $\square$

**Example 4.** Let  $M$  be a manifold and let  $N \subset M$  be a proper submanifold of codimension 1. Let  $\widetilde{M}$  be the universal covering of  $M$  and let  $\widetilde{N} \subset \widetilde{M}$  be the preimage of  $N$ . Then  $\widetilde{N}$  is collared in  $\widetilde{M}$ . We define a tree dual to  $\widetilde{N} \subset \widetilde{M}$ . Its vertices are the components of  $\widetilde{M} - \widetilde{N}$ . Its edges are the components of  $\widetilde{N}$ . We define the endpoints of an edge associated to  $\widetilde{N}_0$  to be the components of  $\widetilde{M} - \widetilde{N}$  containing  $\widetilde{N}_0$  in their closure. Since  $\widetilde{N}$  is collared in  $\widetilde{M}$ , it follows that each edge has two endpoints. Thus, we have defined a graph. Since  $\widetilde{M}$  is simply connected, this graph is a tree. The action of  $\pi_1(M)$  on  $\widetilde{M}$  defines an action of  $\pi_1(M)$  on  $T$ . The quotient is the graph dual to  $N \subset M$ .

*Applications.* As a first application notice that a group acts freely on a simplicial tree if and only if it is a free group. If  $G$  acts freely on a tree, then  $G$  is identified with the fundamental group of the graph  $T/G$ , which is a free group. Conversely, if  $G$  is free on the set  $W$ , then the wedge of circles indexed by  $W$  has fundamental group identified with  $G$ . The group  $G$  acts freely on the universal covering of this wedge, which is a tree.

As our first application we have the famous

**Corollary 3** (Schreier's theorem). *Every subgroup of a free group is free.*

Using the relation between the number of generators of a free group and the Euler characteristic of the wedge of circles, one can also establish the Schreier index formula.

*If  $G$  is a free group of rank  $r$ , and if  $G' \subset G$  is a subgroup of index  $n$ , then  $G'$  is a free group of rank  $n(r - 1) + 1$ .*

One can also use the theory of groups acting on trees to give a generalization of the Kurosh subgroup theorem. Here is the classical statement.

**Theorem** (Kurosh subgroup theorem). *Let  $G = G_1 * G_2$  be a free product. Let  $G' \subset G$  be a subgroup. Suppose that  $G'$  is not decomposable nontrivially as a free product. Then either  $G' \cong \mathbf{Z}$  or  $G'$  is conjugate in  $G$  to a subgroup of either  $G_1$  or  $G_2$ .*

*Proof.* The decomposition  $G = G_1 * G_2$  gives an action of  $G$  on a simplicial tree  $T$  with trivial edge stabilizers and with each vertex stabilizer being conjugate in  $G$  to either  $G_1$  or  $G_2$ . Consider the induced action of  $G'$  on  $T$ . Then  $G'$  has a graph of groups decomposition where all the vertex groups are conjugate in  $G$  to subgroups of either  $G_1$  or  $G_2$  and all the edge groups are trivial. If this decomposition is trivial, then  $G'$  is conjugate to a subgroup of either  $G_1$  or

$G_2$ . If it is nontrivial, then either  $G'$  is a nontrivial free product or  $G'$  has a nontrivial free factor.  $\square$

The generalization which is a natural consequence of the theory of groups acting on trees is:

**Theorem 5.** *Let  $G$  be the fundamental group of a graph of groups, with the vertex groups being  $G_v$  and the edge groups being  $G_e$ . Suppose that  $G' \subset G$  is a subgroup with the property that its intersection with every conjugate in  $G$  of each  $G_e$  is trivial. Then  $G'$  is the fundamental group of a graph of groups all of whose edge groups are trivial. In particular,  $G'$  is isomorphic to a free product of a free group and intersections of  $G$  with various conjugates of the  $G_v$ .*

*Proof.* The restriction of the universal action of  $G$  to  $G'$  produces an action of  $G'$  on a tree with trivial edge stabilizers and with vertex stabilizers exactly the intersections of  $G'$  with the conjugates of the  $G_v$ .  $\square$

**Corollary 6.** *With  $G$  and  $G'$  as in Theorem 5, if  $G'$  is not a nontrivial free product, then it is either isomorphic to  $\mathbf{Z}$  or is conjugate to a subgroup of  $G_v$  for some vertex  $v$ .*

**Corollary 7.** *With  $G$  as in Theorem 5, if  $G' \subset G$  is a subgroup with the property that the intersection of  $G'$  with every conjugate of  $G_v$  in  $G$  is trivial, then  $G'$  is a free group.*

*Proof.* The point stabilizers for the universal action of  $G$  are subgroups of conjugates of the  $G_v$ . Thus, under the hypothesis, the restriction of the universal action to  $G'$  is free.  $\square$

These examples and applications give ample justification for the statement that the theory of groups acting on trees is a natural extension of classical combinatorial group theory.

**The tree associated to  $SL_2$  over a local field.** One of the main reasons that Serre was led to investigate the theory of groups acting on trees was to construct a space to play the role for  $SL_2(\mathbf{Q}_p)$  that the upper half-plane plays for  $SL_2(\mathbf{R})$ . That space is a tree. Here is the outline of Serre's construction.

Let  $K$  be a (commutative) field with a discrete valuation  $v: K^* \rightarrow \mathbf{Z}$ . Recall that this means that  $v$  is a homomorphism from the multiplicative group  $K^*$  of the field onto the integers with the property that

$$v(x + y) \geq \min(v(x), v(y)).$$

By convention we set  $v(0) = +\infty$ . The valuation ring  $\mathcal{O}(v)$  is the ring of  $\{x \in K | v(x) \geq 0\}$ . This ring is a local ring with maximal ideal generated by any  $\pi \in \mathcal{O}(v)$  with the property that  $v(\pi) = 1$ . The quotient  $\mathcal{O}(v)/\pi\mathcal{O}(v)$  is called the residue field  $k_v$  of the valuation.

Let  $W$  be the vector space  $K^2$  over  $K$ . The group  $SL_2(K)$  of  $2 \times 2$ -matrices with entries in  $K$  and determinant 1 is naturally the group of volume-preserving  $K$ -linear automorphisms of  $W$ .

An  $\mathcal{O}(v)$ -lattice in  $W$  is a finitely generated  $\mathcal{O}(v)$ -submodule  $L \subset W$  which generates  $W$  as a vector space over  $K$ . Such a module is a free  $\mathcal{O}(v)$ -module of rank 2. We say that lattices  $L_1$  and  $L_2$  are equivalent (homothetic) if there is  $\alpha \in K^*$  such that  $L_1 = \alpha \cdot L_2$ . We define the set of vertices  $V$  of a graph to be the set of homothety classes of  $\mathcal{O}(v)$ -lattices of  $W$ . We join two vertices corresponding to homothety classes having lattice representatives

$L_1$  and  $L_2$  related in the following way: There is an  $\mathcal{O}(v)$ -basis  $\{e, f\}$  for  $L_1$  so that  $\{\pi e, f\}$  is an  $\mathcal{O}(v)$ -basis for  $L_2$ . It is an easy exercise to show that this defines a simplicial tree on which  $SL_2(K)$  acts. The stabilizer of any vertex is a conjugate in  $GL_2(K)$  of  $SL_2(K)$ . The quotient graph is the interval. A fundamental domain for the action on the tree is the interval connecting the class of the lattice with standard basis  $\{e, f\}$  to the class of the lattice with basis  $\{e, \pi f\}$ . The stabilizer of the edge joining these two lattices is the subgroup

$$\Delta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}(v)) \mid c \in \pi\mathcal{O}(v) \right\}.$$

Thus, we have

$$SL_2(K) \cong SL_2(\mathcal{O}(v)) *_\Delta SL_2(\mathcal{O}(v))'$$

where

$$SL_2(\mathcal{O}(v))' = \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix} SL_2(\mathcal{O}(v)) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}.$$

If  $A \in SL_2(K)$  and if  $[L] \in T$ , then the distance that  $A$  moves  $[L]$  is given as follows. We take an  $\mathcal{O}(v)$ -basis for  $L$  and use it to express  $A$  as a  $2 \times 2$  matrix. The absolute value of the minimum of the valuation of the 4 matrix entries is the distance that  $[L]$  is moved. In particular, if  $\text{trace}(A)$  has negative valuation then  $A$  fixes no point of  $T$ .

*Applications.* In the special case of  $SL_2(\mathbf{Q}_p)$  we have a decomposition

$$SL_2(\mathbf{Q}_p) = SL_2(\mathbf{Z}_p) *_\Delta SL_2(\mathbf{Z}_p)'$$

The maximal compact subgroups of  $SL_2(\mathbf{Q}_p)$  are the conjugates of  $SL_2(\mathbf{Z}_p)$ . We say that a subgroup of  $SL_2(\mathbf{Q}_p)$  is discrete if its intersection with any maximal compact subgroup is finite.

**Corollary 8** (Ihara's Theorem). *Let  $G \subset SL_2(\mathbf{Q}_p)$  be a torsion-free, discrete subgroup. Then  $G$  is a free group.*

*Proof.* If  $G$  is both torsion-free and discrete, then its intersection with any conjugate of  $SL_2(\mathbf{Z}_p)$  is trivial. Applying Corollary 7 yields the result.  $\square$

If  $C$  is a smooth curve, then let  $\mathbf{C}(C)$  denote the field of rational functions on  $C$ . The valuations of  $\mathbf{C}(C)$  are in natural one-to-one correspondence with the points of the completion  $\widehat{C}$  of  $C$ . For each  $p \in \widehat{C}$  the associated valuation  $v_p$  on  $\mathbf{C}(C)$  is given by  $v_p(f)$  is the order of the zero of  $f$  at  $p$ . (If  $f$  has a pole at  $p$ , then by convention the order of the zero of  $f$  at  $p$  is minus the order of the pole of  $f$  at  $p$ .) Let  $H$  be a fixed finitely presented group. The representations of  $H$  into  $SL_2(\mathbf{C})$  form an affine complex algebraic variety, called the representation variety, whose coordinate functions are the matrix entries of generators of  $H$ . Suppose that we have an algebraic curve  $C$  in this variety which is not constant on the level of characters. That is to say there is  $h \in H$  such that the trace of  $\rho(h)$  varies as  $\rho$  varies in  $C$ . Let  $K$  be the function field of this curve. Each ideal point  $p$  of  $C$  is identified with a discrete valuation  $v_p$  of  $K$  supported at infinity in the sense that there is a regular (polynomial) function  $f$  on  $C$  with  $v_p(f) < 0$ . Because of the condition that  $C$  be a nontrivial curve of characters, for some ideal point  $p$ ,

here is  $h \in H$  such that the regular function  $\rho \mapsto \text{tr}_h(\rho) = \text{trace}(\rho(h))$  has negative value under  $v_p$ .

Associated to such a valuation  $v_p$  we have an action of  $SL_2(K)$  on a simplicial tree  $T$ . We also have the tautological representation of  $H$  into  $SL_2(K)$  (in fact into  $SL_2$  of the coordinate ring of regular functions on  $C$ ). (In order to define this representation, notice first that  $C$  is a family of representations of  $H$  into  $SL_2(\mathbb{C})$ . The tautological representation assigns to  $h \in H$  the matrix

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where  $f_{ij}(c)$  is the  $ij$ th entry of  $c(h) \in SL_2(\mathbb{C})$ .) Thus, there is an induced action of  $H$  on a simplicial tree. If  $v_p(\text{tr}_h) < 0$  it follows that the element  $h$  fixes no point of  $T$ . Thus, under our hypotheses on  $C$  and  $v_p$ , it follows that the action of  $H$  on the tree is nontrivial in the sense that  $H$  does not fix a point of the tree.

According to Corollary 2 we have proved

**Corollary 9** ([4]). *Let  $H$  be a finitely presented group. Suppose that the character variety of representations of  $H$  into  $SL_2(\mathbb{C})$  is positive dimensional. Then there is an action of  $H$  on a tree without fixed point. In particular,  $H$  has a nontrivial decomposition as a free product with amalgamation or  $H$  has an HNN-decomposition.*

## 2. $\Lambda$ -TREES

In this section we define  $\Lambda$ -trees as the natural generalization of simplicial trees. We prove the analogue of the result connecting trees and  $SL_2$ . Namely, if  $K$  is a local field with valuation  $v: K^* \rightarrow \Lambda$  and if  $G(K)$  is a rank-1 group, then there is a  $\Lambda$ -tree on which  $G(K)$  acts. We then take up the basics of the way groups act on  $\Lambda$ -trees. We classify single isometries of  $\Lambda$ -trees into three types and use this to define the hyperbolic length function of an action. This leads to a definition of the space of all nontrivial, minimal actions of a given group on  $\Lambda$ -trees. We finish the section with a brief discussion of base change in  $\Lambda$ .

Let us begin by reformulating the notion of a simplicial tree in a way that will easily generalize. The vertices  $V$  of a simplicial tree  $T$  are a set with an integer-valued distance function. Namely, the distance from  $v_0$  to  $v_1$  is the path distance or equivalently the minimal number of edges in a simplicial path in  $T$  from  $v_0$  to  $v_1$ . The entire tree  $T$  can be reconstructed from the set  $V$  and the distance function. The reason is that two vertices of  $T$  are joined by an edge if and only if the distance between them is 1. A question arises as to which integer-valued distance functions arise in this manner from simplicial trees. The answer is not too hard to discover. It is based on the notion of a segment. A  $\mathbb{Z}$ -segment in an integer valued metric space is a subset which is isometric to a subset of the form  $\{t \in \mathbb{Z} | 0 \leq t \leq n\}$ . The integer  $n$  is called the length of the  $\mathbb{Z}$ -segment. The points corresponding to 0 and  $n$  are called the endpoints of the  $\mathbb{Z}$ -segment.

**Theorem 10.** *Let  $(V, d)$  be an integer-valued metric space. Then there is a simplicial tree  $T$  such that the path distance function of  $T$  is isometric to  $(V, d)$  if and only if the following hold*

- (a) For each  $v, w \in V$  there is a  $\mathbf{Z}$ -segment in  $V$  with endpoints  $v$  and  $w$ . This simply means that there is a sequence  $v = v_0, v_1, \dots, v_n = w$  such that for all  $i$  we have  $d(v_i, v_{i+1}) = 1$ .
- (b) The intersection of two  $\mathbf{Z}$ -segments with an endpoint in common is a  $\mathbf{Z}$ -segment.
- (c) The union of two  $\mathbf{Z}$ -segments in  $V$  whose intersection is a single point which is an endpoint of each is itself a  $\mathbf{Z}$ -segment.

Given a set with such an integer-valued distance function, one constructs a graph by connecting all pairs of points at distance one from each other. Condition (a) implies that the result is a connected graph. Condition (b) implies that it has no loops. Condition (c) implies that the metric on  $V$  agrees with the path metric on this tree.

This definition can be generalized by replacing  $\mathbf{Z}$  by any (totally) ordered abelian group  $\Lambda$ . Before we do this, let us make a couple of introductory remarks about ordered abelian groups. An ordered abelian group is an abelian group  $\Lambda$  which is partitioned in three subsets  $P, N, \{0\}$  such that for each  $x \neq 0$  we have exactly one of  $x, -x$  is contained in  $P$  and with  $P$  closed under addition.  $P$  is said to be the set of *positive* elements. We say that  $x > y$  if  $x - y \in P$ . A *convex subgroup* of  $\Lambda$  is a subgroup  $\Lambda_0$  with the property that if  $y \in \Lambda_0 \cap P$  and if  $0 < x < y$  then  $x \in \Lambda_0$ . The *rank* of an ordered abelian group  $\Lambda$  is one less than the length of the maximal chain of convex subgroups, each one proper in the next. An ordered abelian group has rank 1 if and only if it is isomorphic to a subgroup of  $\mathbf{R}$ .

A  $\Lambda$ -metric space is a pair  $(X, d)$  where  $X$  is a set and  $d$  is a  $\Lambda$ -distance function  $d: X \times X \rightarrow \Lambda$  satisfying the usual metric axioms. In  $\Lambda$  there are segments  $[a, b]$  given by  $\{\lambda \in \Lambda \mid a \leq \lambda \leq b\}$ . More generally, in a  $\Lambda$ -metric space a *segment* is a subset isometric to some  $[a, b] \subset \Lambda$ . It is said to be *nondegenerate* if  $a < b$ . As before, each nondegenerate segment has two endpoints.

**Definition** ([24], [13]). A  $\Lambda$ -tree is a  $\Lambda$ -metric space  $(T, d)$  such that:

- (a) For each  $v, w \in T$  there is a  $\Lambda$ -segment in  $T$  with endpoints  $v$  and  $w$ .
- (b) The intersection of two  $\Lambda$ -segments in  $T$  with an endpoint in common is a  $\Lambda$ -segment.
- (c) The union of two  $\Lambda$ -segments of  $T$  whose intersection is a single point which is an endpoint of each is itself a  $\Lambda$ -segment.

**Example.** Let  $v: K^* \rightarrow \Lambda$  be a possibly nondiscrete valuation. Of course, by definition the value group  $\Lambda$  is an ordered abelian group. Associated to  $SL_2(K)$  there is a  $\Lambda$ -tree on which it acts. The points of this  $\Lambda$ -tree are again homothety classes of  $\mathcal{O}(v)$ -lattices in  $K^2$ . The  $\Lambda$ -distance between two homothety classes of lattices is defined as follows. Given the classes there are representative lattices  $L_0 \subset L_1$  with quotient  $L_0/L_1$  and  $\mathcal{O}(v)$ -module of the form  $\mathcal{O}(v)/\alpha\mathcal{O}(v)$  for some  $\alpha \in \mathcal{O}(v)$ . The distance between the classes is  $v(\alpha)$ . In particular, the stabilizers of various points will be conjugates of  $SL_2(\mathcal{O}(v))$ . Thus, every element  $\gamma \in SL_2(K)$  which fixes a point in this  $\Lambda$ -tree has  $v(\text{tr}(\gamma)) \geq 0$ . If  $\rho: H \rightarrow SL_2(K)$  is a representation, then there is an induced action of  $H$  on the  $\Lambda$ -tree. This action will be without fixed point for the whole group if there is  $h \in H$  such that  $v(\text{tr} \rho(h)) < 0$ .

**The tree associated to a semisimple rank-1 algebraic group over a local field.** The example in the previous section showed how  $SL_2$  over a local field with value group  $\Lambda$  gives rise to a  $\Lambda$ -tree. In fact this construction generalizes to any semisimple real rank-1 group. We will content ourselves with considering the case of  $SO(n, 1)$ . Let  $K$  be a field with a valuation  $v: K^* \rightarrow \Lambda$ . We suppose that the residue field  $k_v$  is formally real in the sense that  $-1$  is not a sum of squares in  $k_v$ . (In particular,  $k_v$  is of characteristic zero.) Let  $q: K^{n+1} \rightarrow K$  be the standard quadratic form of type  $(n, 1)$ ; i.e.,

$$q(x_0, \dots, x_n) = x_0x_1 + x_2^2 + \dots + x_n^2.$$

We denote by  $\langle \cdot, \cdot \rangle$  the induced bilinear form.

Let  $SO_K(n, 1) \subset SL_{n+1}(K)$  be the automorphism group of  $q$ . By a unimodular  $\mathcal{O}(v)$ -lattice in  $K^{n+1}$  we mean a finitely generated  $\mathcal{O}(v)$ -module which generates  $K^{n+1}$  over  $K$  and for which there is a standard  $\mathcal{O}(v)$ -basis, i.e., an  $\mathcal{O}(v)$ -basis  $\{e_0, \dots, e_n\}$  with  $q(\sum y_i e_i) = y_0y_1 + y_2^2 + \dots + y_n^2$ . Let  $T$  be the set of unimodular  $\mathcal{O}(v)$ -lattices. To define a  $\Lambda$ -metric on  $T$  we need the following lemma which is proved directly.

**Lemma 11.** *If  $L_0$  and  $L_1$  are unimodular  $\mathcal{O}(v)$ -lattices, then there is a standard  $\mathcal{O}(v)$ -basis for  $L_0$ ,  $\{e_0, e_1, \dots, e_n\}$  and  $\alpha \in \mathcal{O}(v)$  such that*

$$\{\alpha e_0, \alpha^{-1} e_1, e_2, \dots, e_n\}$$

*is a standard basis for  $L_1$ .*

We then define the distance between  $L_0$  and  $L_1$  to be  $v(\alpha)$ . It is not too hard to show that this space of unimodular  $\mathcal{O}(v)$ -lattices with this  $\Lambda$ -distance function forms a  $\Lambda$ -tree (see [11]). Notice that the  $\Lambda$ -segment between  $L_0$  and  $L_1$  is defined by taking the unimodular  $\mathcal{O}(v)$ -lattices with bases  $\{\beta e_0, \beta^{-1} e_1, e_2, \dots, e_n\}$  as  $\beta$  ranges over elements of  $\mathcal{O}(v)$  with  $v(\beta) \leq v(\alpha)$ . (This then generalizes the construction in Example 1 to  $SO(n, 1)$  and we repeat to the other rank-1 groups.)

The tree associated to  $SO(n, 1)$  over a local field  $K$  will be a simplicial tree exactly when the valuation on  $K$  is discrete (i.e., when the value group is  $\mathbf{Z}$ ).

**Basics of group actions on  $\Lambda$ -trees** (cf. [13, 1, 3]). We describe some of the basic results about the way groups act on  $\Lambda$ -trees. The statements and proofs are all elementary and directly "tree-theoretic." Let us begin by classifying a single automorphism  $\alpha$  of a  $\Lambda$ -tree  $T$ . There are three cases:

- (1)  $\alpha$  has a fixed point in  $T$ . Then the fixed point set  $F_\alpha$  of  $\alpha$  is a subtree, and any point not in the fixed point set is moved by  $\alpha$  a distance equal to twice the distance to  $F_\alpha$ . The midpoint of the segment joining  $x$  to  $\alpha(x)$  is contained in  $F_\alpha$ . (See Figure 4.)
- (2) There is a segment of length  $\lambda \in \Lambda$  but  $\lambda \notin 2\Lambda$  which is flipped by  $\alpha$ . (See Figure 5.)
- (3) There is an axis for  $\alpha$ ; that is to say there is an isometry from a convex subgroup of  $\Lambda$  into  $T$  whose image is invariant under  $\alpha$  and on which  $\alpha$  acts by translation by a positive amount  $\tau(\alpha)$ .

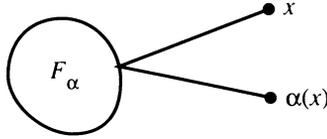


FIGURE 4

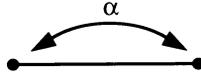


FIGURE 5

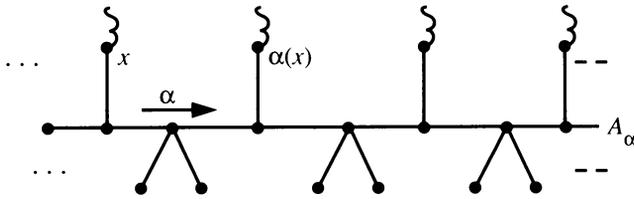


FIGURE 6

The set of all points in  $T$  moved by  $\alpha$  this distance  $\tau(\alpha)$  themselves form the maximal axis  $A_\alpha$  for  $\alpha$ . Any other point  $x \in T$  is moved by  $\alpha$  a distance  $2d(x, A_\alpha) + \tau(\alpha)$  where  $d(x, A_\alpha)$  is the distance from  $x$  to  $A_\alpha$ . (See Figure 6.)

The *hyperbolic length*  $l(\alpha)$  of  $\alpha$  is said to be 0 in the first two cases and  $\tau(\alpha)$  in the last case. The automorphism  $\alpha$  is said to be *hyperbolic* if  $\tau(\alpha) > 0$ , *elliptic* if  $\alpha$  has a fixed point and to be an *inversion* in the remaining case. The *characteristic set*  $A_\alpha$  of  $\alpha$  is  $F_\alpha$  in Case 1,  $A_\alpha$  in Case 3, and empty in Case 2. Thus,  $C_\alpha$  is the set of points of  $T$  which are moved by  $\alpha$  a distance equal to  $l(\alpha)$ . Clearly, there are no inversions if  $2\Lambda = \Lambda$ .

Notice that if  $x \notin C_\alpha$ , then the segment  $S$  joining  $x$  to  $\alpha(x)$  has the property that  $S \cap \alpha(S)$  is a segment of positive length. We denote this by saying that the direction from  $\alpha(x)$  toward  $x$  and the direction from  $\alpha(x)$  toward  $\alpha^2(x)$  agree. Here is a lemma which indicates the ‘tree’ nature of  $\Lambda$ -trees.

**Lemma 12.** *Suppose that  $g, h$  are isometries of a  $\Lambda$ -tree  $T$  with the property that  $g, h$ , and  $gh$  each have a fixed point in  $T$ . Then there is a common fixed point for  $g$  and  $h$ .*

Notice that this result is false for the plane: two rotations of opposite angle about distinct points in the plane fail to satisfy this lemma.

*Proof.* Suppose that  $g$  and  $h$  have fixed points but that  $F_g \cap F_h = \emptyset$ . Then there is a bridge between  $F_g$  and  $F_h$ , i.e., a segment which meets  $F_g$  in one end and  $F_h$  in the other. Let  $x$  be the initial point of this bridge. Then the direction from  $gh(x)$  toward  $x$  and the direction from  $gh(x)$  toward  $(gh)^2(x)$  are distinct. Thus,  $x \in C_{gh}$ . Since  $x$  is not fixed by  $gh$ , it follows that  $gh$  is hyperbolic. (See Figure 7 on page 100.)  $\square$

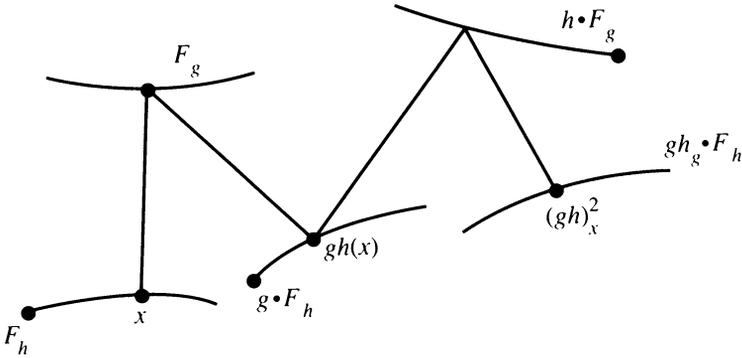


FIGURE 7

**Corollary 13.** *If  $G$  is a finitely generated group of automorphisms of a  $\Lambda$ -tree such that each element in  $G$  is elliptic, then there is a point of  $T$  fixed by the entire group  $G$ . In particular, if  $2\Lambda = \Lambda$  and if the hyperbolic length function of the action is trivial, then there is a point of the tree fixed by the entire group.*

Let us now describe some of the basic terminology in the theory of  $\Lambda$ -trees. Let  $T$  be a  $\Lambda$ -tree. A *direction* from a point  $x \in t$  is the germ of a nondegenerate  $\Lambda$ -segment with one endpoint being  $x$ . The point  $x$  is said to be a *branch point* if there are at least three distinct directions from  $x$ . It is said to be a *dead end* if there is only one direction. Otherwise,  $x$  is said to be a *regular point*.

If  $T$  has a minimal action of a countable group  $G$ , then  $T$  has no dead ends and only countable many branch points. Each branch point has at most countably many directions. It may well be the case when  $\Lambda = \mathbf{R}$  that the branch points are dense in  $T$ .

If  $G \times T \rightarrow T$  is an action of a group on a  $\Lambda$ -tree, then the function associating to each  $g \in G$ , its hyperbolic length, is a class function in the sense that it is constant on each conjugacy class. We denote by  $\mathcal{C}$  the set of conjugacy classes in  $G$ . We define the *hyperbolic length function* of an action of  $G$  on a  $\Lambda$ -tree to be

$$l: \mathcal{C} \rightarrow \Lambda^{\geq 0}$$

which assigns to each  $c \in \mathcal{C}$  the hyperbolic length of any element of  $G$  in the class  $c$ .

**The space of actions** (cf. [3]). Several natural questions arise. To what extent does the hyperbolic length function determine the action? Which functions are hyperbolic length functions of actions? The second question has been completely answered (at least for subgroups of  $\mathbf{R}$ ). There are some obvious necessary conditions (each condition being an equation or weak inequality between the hyperbolic length of finitely many group elements), first laid out in [3]. In [19] it was proved that these conditions characterize the set of hyperbolic length functions.

Let us consider the first question. The idea is that the hyperbolic length function of an action should be like the character of a representation. There are a couple of hurdles to surmount before this analogy can be made precise. First of all, the inversions cause problems much like they do in the simplicial

case. Thus, as in the simplicial case, one restricts to actions without inversions (which is no restriction at all in the case  $\Lambda = \mathbf{R}$ ). One operation which changes the action but not its hyperbolic length function is to take a subtree. For most actions there is a unique minimal invariant subtree. If the group is finitely generated, the actions which do not necessarily have a unique invariant subtree are those that fix points on the tree. These we call *trivial actions*. It is natural to restrict to nontrivial actions and to work with the minimal invariant subtree. In this context the question then becomes to what extent the minimal invariant subtree of a nontrivial action is determined by the hyperbolic length function of the action. There is a special case of ‘reducible-type’ actions much like the case of reducible representations where the hyperbolic length function does not contain all the information about the minimal invariant subtree, but this case is the exception. For all other actions one can reconstruct the minimal action from the hyperbolic length function. These considerations lead to a space of nontrivial, minimal actions of a given finitely presented group on  $\Lambda$ -trees. It is the space of hyperbolic length functions. When  $\Lambda$  has a topology, e.g., if  $\Lambda \subset \mathbf{R}$  then this space of actions inherits a topology from the natural topology on the set of  $\Lambda$ -valued functions on the group. For example, when  $\Lambda = \mathbf{R}$  the space of nontrivial, minimal actions is closed subspace of  $(\mathbf{R}^{\geq 0})^{\mathcal{G}} - \{0\}$ . It is natural to divide this space by the action of  $\mathbf{R}^+$  by homotheties forming a projective space

$$P((\mathbf{R}^{\geq 0})^{\mathcal{G}}) = ((\mathbf{R}^{ge0})^{\mathcal{G}} - \{0\}/\mathbf{R}^+).$$

This projective space is compact, and the space of projective classes of actions (or projectivized hyperbolic length functions) is a closed subset of this projective space

$$\mathcal{P}A(G) \subset P((\mathbf{R}^{\geq 0})^{\mathcal{G}}).$$

**Base change** (cf. [1]). Suppose that  $\Lambda \subset \Lambda'$  is an inclusion of ordered abelian groups. Suppose that  $T$  is a  $\Lambda$ -tree. Then there is an extended tree  $T \otimes_{\Lambda} \Lambda'$ . In brief one replaces each  $\Lambda$ -segment in  $T$  with a  $\Lambda'$ -segment. One example of this is  $T \otimes_{\mathbf{Z}} \mathbf{R}$ . This operation takes a  $\mathbf{Z}$  tree (which is really the set of vertices of a simplicial tree) and replaces it with the  $\mathbf{R}$ -tree, which is the geometric realization of the simplicial tree. The operation  $T \otimes_{\mathbf{Z}} \mathbf{Z}[1/2]$  is the operation of barycentric subdivision. Base change does not change the hyperbolic length function.

If  $G$  acts on a  $\Lambda$ -tree  $T$ , then it acts without inversions on the  $\Lambda[1/2]$ -tree  $T \otimes_{\Lambda} \Lambda[1/2]$ .

The operation of base change is a special case of a more general construction that embeds  $\Lambda$ -metric spaces satisfying a certain 4-point property isometrically into  $\Lambda$ -trees; see [1].

Finally, if  $\Lambda'$  is a quotient of  $\Lambda$  by a convex subgroup  $\Lambda_0$ , then there is an analogous quotient operation that applies to any  $\Lambda$  tree to produce a  $\Lambda'$ -tree as quotient. The fibers of the quotient map are  $\Lambda_0$ -trees.

### 3. COMPACTIFYING THE SPACE OF CHARACTERS

Let  $G$  be a finitely presented group.  $R(G) = \text{Hom}(G, SO(n, 1))$  is naturally the real points of an affine algebraic variety defined over  $\mathbf{Z}$ . In fact, if  $\{g_1, \dots, g_k\}$  are generators for  $G$  then we have

$$\text{Hom}(G, SO(n, 1)) \subset SO(n, 1)^k \subset M(n \times n)^k = \mathbf{R}^{kn^2}$$

is given by the polynomial equations which say that (1) all the  $g_i$  are mapped to elements of  $SO(n, 1)$  and (2) that all the relations among the  $\{g_i\}$  which hold in  $G$  hold for their images in  $SO(n, 1)$ . We denote by  $SO_{\mathbf{C}}(n, 1)$  and  $R_{\mathbf{C}}(G)$  the complex versions of these objects. Each  $g \in G$  determines  $n^2$  polynomial functions on  $R(G)$ . These functions assign to each representation the matrix entries of the representation on the given element  $g$ . The coordinate ring for  $R(G)$  is generated by these functions. These is an action of  $SO_{\mathbf{C}}(n, 1)$  and  $R_{\mathbf{C}}(G)$  by conjugation. The quotient affine algebraic variety is called the character variety and is denoted  $\chi_{\mathbf{C}}(G)$ . Though it is a complex variety, it is defined over  $\mathbf{R}$ . Its set of real points,  $\chi(G)$ , is the equivalence classes of complex representations with real characters. The polynomial functions on  $\chi(G)$  are the polynomial functions on  $R(G)$  invariant by conjugation. In particular the traces of the various elements of  $G$  are polynomials on  $\chi(G)$ . The character variety contains a subspace  $Z(G)$  of equivalence classes of real representations. The subspace  $Z(G)$  is a semialgebraic subset of  $\chi(G)$  and is closed in the classical topology. Each fiber of the map  $R(G) \rightarrow Z(G)$  either is made up of representations whose images are contained in parabolic subgroups of  $SO(n, 1)$  or is made up of finitely many  $SO(n, 1)$ -conjugacy classes of representations into  $SO(n, 1)$ .

The variety  $\chi(G)$  and the subspace  $Z(G)$  are usually not compact. Our purpose here is produce natural compactifications of  $Z(G)$  and  $\chi(G)$  as topological spaces and to interpret the ideal points at infinity. The idea is to map the character variety into a projective space whose homogeneous coordinates are indexed by  $\mathcal{E}$ , the set of conjugacy classes in  $G$ . The map sends a representation to the point whose homogeneous coordinates are the logs of the absolute values of its traces on the conjugacy classes. The first result is that the image of this map has compact closure in the projective space. As we go off to infinity in the character variety at least one of these traces is going to infinity. Thus, the point that we converge to in the projective space measures the relative growth rates of the logs of the traces of the various conjugacy classes. It turns out that any such limit point in the projective space can be described by a valuation supported at infinity on the character variety (or on some subvariety of it). These then can be reinterpreted in terms of actions of  $G$  on  $\mathbf{R}$ -trees. This then is the statement for the case of  $SO(n, 1)$ : A sequence of representations of  $G$  into  $SO(n, 1)$  which has unbounded characters has a subsequence that converges to an action of  $G$  on an  $\mathbf{R}$ -tree. This material is explained in more detail in [13] and [11].

**The projective space.** We begin by describing the projective space in which we shall work. Let  $\mathcal{E}$  denote the set of conjugacy classes in  $G$ . We denote by  $P(\mathcal{E})$  the projective space

$$((\mathbf{R}^{\geq 0})^{\mathcal{E}} - \{0\})/\mathbf{R}^+$$

where  $\mathbf{R}^+$  acts by homotheties. Thus, a point in  $P(\mathcal{E})$  has homogeneous coordinates  $[x_\gamma]_{\gamma \in \mathcal{E}}$  with the convention that each  $x_\gamma$  is a nonnegative number.

**The main result.** Before we broach all the technical details, let us give a consequence which should serve to motivate the discussion.

**Theorem 14.** *Let  $\rho_k: G \rightarrow SO(n, 1)$  be a sequence of representations with the property that for some  $g \in G$  we have  $\{\text{tr}(\rho_k(g))\}_k$  is unbounded. Then after*

replacing the  $\{\rho_k\}$  with a subsequence we can find a nontrivial action

$$\varphi: G \times T \rightarrow T$$

of  $G$  on an  $\mathbf{R}$ -tree such that the positive part of the logs of the absolute value of the traces of the  $\rho_k(g)$  converge projectively to the hyperbolic length function of  $\varphi$ ; i.e., so that if

$$p_k = [\max(0, \log |\operatorname{tr}(\rho_k(\gamma))|)]_{\gamma \in \mathcal{E}} \in P(\mathcal{E})$$

then

$$\lim_{k \rightarrow \infty} p_k = [l(\varphi(\gamma))]_{\gamma \in \mathcal{E}}.$$

If all the representations  $\rho_k$  of  $G$  into  $SO(n, 1)$  are discrete and faithful, then the limiting action of  $G$  on an  $\mathbf{R}$ -tree has the property that for any nondegenerate segment  $J \subset T$  the stabilizer of  $J$  under  $\varphi$  is a virtually abelian group.

The component of the identity  $SO^+(n, 1)$  of  $SO(n, 1)$  is an isometry group of hyperbolic  $n$ -space. We define the hyperbolic length of  $\alpha \in SO(n, 1)$  to be the minimum distance a point in hyperbolic space is moved by  $\alpha$ . The quantity  $\max(0, \log(|\operatorname{tr}(\alpha)|))$  differs from the hyperbolic length of  $\alpha$  by an amount bounded independent of  $\alpha$ . Expressed vaguely, the above result says that as actions of  $G$  on hyperbolic  $n$ -space degenerate the hyperbolic lengths of these actions, after rescaling, converge to the hyperbolic length of an action of  $G$  on an  $\mathbf{R}$ -tree.

The rest of this section is devoted to indicating how one establishes this result.

**Mapping valuations into  $P(\mathcal{E})$ .** Let us describe how valuations on the function field of  $R(G)$  determine points of  $P(\mathcal{E})$ . Suppose that  $\Lambda \subset \mathbf{R}$ . Then any collection of elements  $\{x_\gamma\}_{\gamma \in \mathcal{E}}$  of  $\Lambda^{\geq 0}$ , not all of which are zero, determine an element of  $P(\mathcal{E})$ . This can be generalized to any ordered abelian group of finite rank. Suppose that  $\Lambda$  is an ordered abelian group and that  $x, y \in \Lambda$  are nonnegative elements, not both of which are zero. Then there is a well-defined ratio  $x/y \in \mathbf{R}^{\geq 0} \cup \infty$ . More generally, if  $\{x_\gamma\}_{\gamma \in \mathcal{E}}$  is a collection of elements in  $\Lambda$  with the property that  $x_\gamma \geq 0$  for all  $\gamma \in \mathcal{E}$  and that  $x_\gamma > 0$  for some  $\gamma \in \mathcal{E}$  and if  $\Lambda$  is of finite rank, then we have a well-defined point

$$[x_\gamma]_{\gamma \in \mathcal{E}} \in P(\mathcal{E}).$$

Now suppose that  $K$  is the function field of  $R(G)$ . This field contains the trace functions  $\operatorname{tr}_\gamma$  for all  $\gamma \in \mathcal{E}$ . A sequence in  $\chi(G)$  goes off to infinity if and only if at least one of the traces  $\operatorname{tr}_\gamma$  is unbounded on the sequence.

**Lemma 15.** *Let  $v: K^* \rightarrow \Lambda$  is a valuation, trivial on the constant functions, which is supported at infinity in the sense that for some  $\gamma \in \mathcal{E}$  we have  $v(\operatorname{tr}_\gamma) < 0$ . Then  $\Lambda$  is of finite rank. Thus we can define a point  $\mu(v)$  in  $P(\mathcal{E})$  by*

$$\mu(v) = [\max(0, -v(\operatorname{tr}_\gamma))]_{\gamma \in \mathcal{E}}.$$

*Similarly, if  $X \subset R(G)$  is a subvariety defined over  $\mathbf{Q}$  and if  $v$  is a valuation on its function field  $K_X$ , trivial on the constant functions, supported at infinity then we can define  $\mu(v)$  to be the same formula.*

The reason that we take  $\max(0, -v(\operatorname{tr}_\gamma))$  is that this number is the logarithmic growth of  $\operatorname{tr}_\gamma$  as measured by  $v$ . For example, if the field is the function field of a curve and  $v$  is a valuation supported at an ideal point  $p$  of the curve then  $\max(0, -v(f))$  is exactly the order of pole of  $f$  at  $p$ .

**How actions of  $G$  on  $\Lambda$ -trees determine points in  $P(\mathcal{E})$ .** Another source of points in  $P(\mathcal{E})$  is nontrivial actions of  $G$  on  $\Lambda$ -trees, once again for  $\Lambda$  an ordered abelian group of finite rank. Let  $\varphi: G \times T \rightarrow T$  be an action of  $G$  on a  $\Lambda$ -tree. Let  $T'$  be the  $\Lambda[1/2]$ -tree  $T \otimes_{\Lambda} \Lambda[1/2]$ . Then there is an extended action of  $G$  on  $T'$  which is without inversions. Since  $G$  is finitely generated, this action is nontrivial if and only if its hyperbolic length function  $l: \mathcal{E} \rightarrow \Lambda^{\geq 0}$  is nonzero. If the action is nontrivial, then we have the point

$$l(\varphi) = [l(\varphi(\gamma))]_{\gamma \in \mathcal{E}} \in P(\mathcal{E}).$$

Actually, it is possible to realize the same point in  $P(\mathcal{E})$  by an action of  $G$  on an  $\mathbf{R}$ -tree. The reason is that we can find a convex subgroup  $\Lambda_0 \subset \Lambda$  which contains all the hyperbolic lengths and is minimal with respect to this property. Let  $\Lambda_1 \subset \Lambda_0$  be the maximal proper convex subgroup of  $\Lambda_0$ . Then the  $T$  admits a  $G$ -invariant  $\Lambda_0$ -subtree. This tree has a quotient  $\Lambda_0/\Lambda_1$ -tree on which  $G$  acts. Since  $\Lambda_0/\Lambda_1$  is of rank 1, it embeds as a subgroup of  $\mathbf{R}$ . Thus, by base change we can extend the  $G$  action on this quotient tree to a  $G$  action on  $\mathbf{R}$ . This action determines the same point in  $P(\mathcal{E})$ .

As we saw in the last section, the projective space  $\mathcal{P}A(G)$  of nontrivial minimal actions of  $G$  on  $\mathbf{R}$ -trees sits by definition  $P(\mathcal{E})$ . What we have done here is to show that any action of  $G$  on a  $\Lambda$ -tree, for  $\Lambda$  an ordered abelian group of finite rank, also determines a point of  $\mathcal{P}A(G) \subset P(\mathcal{E})$ .

There is a relationship between these two constructions of points in  $P(\mathcal{E})$ , one from valuations on the function field of  $\chi(G)$  and the other from actions on trees.

**Theorem 16.** *Suppose that  $X \subset R(G)$  is a subvariety defined over  $\mathbf{Q}$  and whose projection to  $\chi(G)$  is unbounded. Suppose that  $K_X$  is its function field and that  $v: K_X^* \rightarrow \Lambda$  is a valuation, trivial on the constant functions, which is supported at infinity and which has formally real residue field. Then associated to  $v$  is an action of  $SO_K(n, 1)$  on a  $\Lambda$ -tree. We have the tautological representation of  $G$  into  $SO_K(n, 1)$  and hence there is an induced action  $\varphi$  of  $G$  on a  $\Lambda$ -tree. The image in  $P(\mathcal{E})$  of the valuation and of this action of  $G$  on the tree are the same; i.e.,  $\mu(v) = l(\varphi)$ .*

**Embedding the character variety in  $P(\mathcal{E})$ .** As we indicated in the beginning of this section, the purpose for introducing the maps from valuations and actions on trees to  $P(\mathcal{E})$  is to give representatives for the ideal points of a compactification of the character variety,  $\chi(G)$ . Toward this end we define a map  $\theta: \chi(G) \rightarrow P(G)$  by

$$\theta([\rho]) = [\max(0, \log(|\text{tr}_{\gamma}(\rho)|))]_{\gamma \in \mathcal{E}}.$$

The map  $\theta$  measures the relative sizes of the logs of the absolute values of the traces of the images of the various conjugacy classes under the representation. It is an elementary theorem (see [13]) that the image  $\theta(\chi(G))$  has compact closure. Thus, there is an induced compactification  $\bar{\chi}(G)$ . The set of ideal points of this compactification, denoted by  $B(\chi(G))$ , is the compact subset of points in  $P(\mathcal{E})$  which are limits of sequences  $\{\theta([\rho_i])\}_i$  where  $\{[\rho_i]\}_i$  is an unbounded sequence in  $\chi(G)$ . We denote by  $B(Z(G))$  the intersection of the closure of  $Z(G)$  with  $B(\chi(G))$ . Here are the two main results that were established in [13] and [11].

**Theorem 17.** *For each point  $b \in B(\chi(G))$  there exist a subvariety  $X$  of  $R(G)$  defined over  $\mathbf{Q}$  and a valuation  $v$  on  $K_X$ , trivial on the constant functions, supported at infinity such that  $\mu(v) = b$ .*

**Theorem 18.** *For each point  $b \in B(Z(G))$  the valuation  $v$  in Theorem 17 with  $\mu(v) = b$  can be chosen to have formally real residue field. Thus, there is a nontrivial action  $\varphi: G \times T \rightarrow T$  on an  $\mathbf{R}$ -tree such that  $b = l(\varphi)$ . Said another way  $B(Z(G))$  is a subset of  $\mathcal{P}A(G) \subset P(\mathcal{E})$ . Lastly, if the point  $b \in B(Z(G))$  is the limit of discrete and faithful representations, then the action  $\varphi$  has the property that the stabilizer of any nondegenerate segment in the tree is a virtually abelian subgroup of  $G$ .*

Theorem 14 is now an immediate consequence of this result.

There is an obvious corollary.

**Corollary 19.** *Let  $G$  be a finitely presented group nonvirtually abelian group. If there is no nontrivial action of  $G$  on an  $\mathbf{R}$ -tree in which the stabilizer of each nondegenerate segment is virtually abelian, then the space of conjugacy classes of discrete and faithful representations of  $G$  into  $SO(n, 1)$  is compact.*

What is not clear in this result, however, is the meaning of the condition about  $G$  admitting no nontrivial actions on  $\mathbf{R}$ -trees with all nondegenerate segments having virtually abelian stabilizers. This is a question to which we shall return in §5.

One thing is clear from the algebraic discussion, however. The discrete valuations of a field are dense in the space of all valuations on the field. Thus, it turns out that a dense subset of  $B(\chi(G))$  is represented by discrete valuations. This leads to the following result.

**Proposition 20.**  *$B(Z(G))$  contains a countable dense subset represented by nontrivial actions of  $G$  on simplicial trees.*

From this result and Corollary 2 we have

**Corollary 21.** *Let  $G$  be a finitely generated group. If  $Z(G)$  is not compact; i.e., if there is a sequence of representations  $\{\rho_k\}_k$  of  $G$  into  $SO(n, 1)$  such that for some  $\gamma \in G$  the sequences of traces  $\{\text{tr}(\rho_k(\gamma))\}_k$  is unbounded, then  $G$  has either a nontrivial free product with amalgamation decomposition or  $G$  has an HNN-decomposition.*

**Geometric approach.** What we have given here is an algebraic approach to establish the degeneration of actions of  $G$  on hyperbolic space to actions of  $G$  on  $\mathbf{R}$ -trees. There is, however, a purely geometric approach to the same theory. In a geometric sense one can take limits of hyperbolic space with a sequence of rescaled metrics so that the curvature goes to  $-\infty$ . Any such limit is an  $\mathbf{R}$ -tree. Given a finitely generated group, there is a finite set of elements  $\{g_1, \dots, g_t\}$  in  $G$  such that for any  $g \in G$  there is a polynomial  $p_g$  in  $t$  variables such that for any representation  $\rho: G \rightarrow SO(n, 1)$  the trace of  $\rho(g)$  is bounded by the value  $p_g(\rho(g_1), \dots, \rho(g_t))$ . Thus, given a sequence of representations  $\rho_k$  then for each  $k$  we replace hyperbolic space by a constant  $\lambda_k$  so that the maximum of the hyperbolic lengths of  $\rho_k(g_1), \dots, \rho_k(g_t)$  is 1. Any limit of

these rescaled hyperbolic spaces will be an  $\mathbf{R}$ -tree on which there is an action of  $G$ . For more details see [12, §§8–10].

#### 4. TREES AND CODIMENSION-1 MEASURED LAMINATIONS

In this section we shall explain the relation between trees and codimension-1 transversely measured laminations. This relationship generalizes Example 4 of §1 where it was shown that if  $N \subset M$  is a compact codimension-1 submanifold, then there is a dual action of  $\pi_1(M)$  on a simplicial tree.

**Definition.** Let  $M$  be a manifold. A (codimension-1 transversely) measured lamination  $(\mathcal{L}, \mu)$  in  $M$  consists of

- (a) a closed subset  $|\mathcal{L}| \subset M$  called the *support of the lamination*  $\mathcal{L}$ ,
- (b) a covering of  $|\mathcal{L}|$  by open subsets  $V$  of  $M$ , called *flow boxes*, which have topological product structures  $V = U \times (a, b)$  (where  $(a, b)$  denotes an open interval) so that  $|\mathcal{L}| \cap V$  is of the form  $U \times X$  where  $X \subset (a, b)$  is a closed subset, and
- (c) for each open set as in (b) a Borel measure on the interval  $(a, b)$  with support equal to  $X \subset (a, b)$ .

The open sets and measures are required to satisfy a compatibility condition. We define the local leaves of the lamination in  $V = U \times (a, b)$  to be the slices  $U \times \{x\}$  for  $x \in X$ . The first compatibility condition is that the germs of local leaves in overlapping flow boxes agree. The second compatibility condition involves transverse paths. A path in a flow box is said to *transverse to the lamination* if it is transverse to each local leaf. Then the measures in the flow box can be integrated over transverse paths in that flow box to give a total measure. If a transverse path lies in the intersection of two flow boxes then the total measures that are assigned to it in each flow box are required to agree.

Two sets of flow boxes covering  $|\mathcal{L}|$  define the same structure if their union forms a compatible system of flow boxes. Equivalently, we can view a measured lamination as a maximal family of compatible flow boxes.

We define an equivalence relation on  $|\mathcal{L}|$ . This equivalence relation is generated by saying that two points are equivalent if they both lie on the same local leaf in some flow box. The equivalence classes are called the *leaves* of the measured lamination. Each leaf is a connected codimension-1 submanifold immersed in a 1-to-1 fashion in  $M$ . It meets each flow box in a countable union of local leaves for that flow box. (See Figure 8.)

If the ambient manifold is compact, then the cross section of the support of a measured lamination is the union of a cantor set where the measure is diffuse,

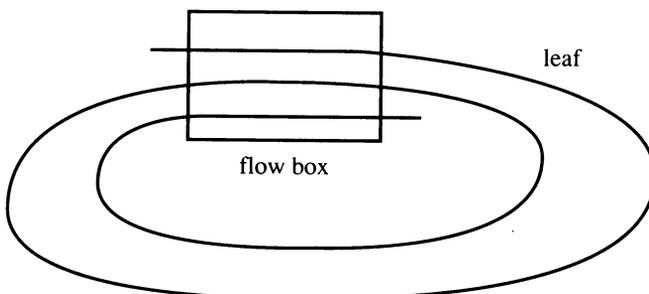


FIGURE 8

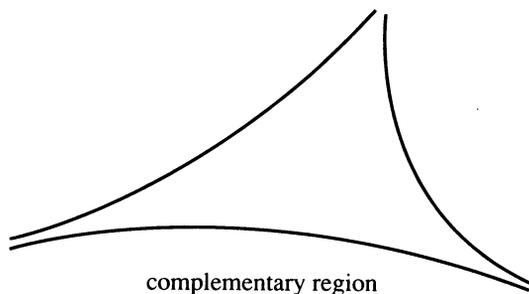


FIGURE 9

an isolated set where the measure has  $\delta$ -masses and nondegenerate intervals. If the lamination has no compact leaves, then every cross section meets the support in a cantor set.

The complement of the support of a measured lamination is an open subset of  $M$ . It consists then of at most countably many components. It turns out that when  $M$  is compact, there can be countably many ‘thin’ or product components bounded by two parallel leaves. The other complementary components are finite in number and are called the ‘big’ complementary regions. (See Figure 9.)

**Example 1.** A compact, codimension-1 submanifold  $N \subset M$  is a codimension-1 measured lamination; we assign the counting measure, that is to say a  $\delta$ -mass of total mass 1 transverse to each component of  $N$ .

**Example 2.** Thurston [23] has considered measured laminations on a closed hyperbolic surface a genus  $g$  all of whose leaves are geodesics. He has shown that the space of all such measured laminations with the topology induced from the weak topology on measures is a real vector space of dimension  $6g - 6$ . The typical geodesic measured lamination has  $4g - 4$  complementary regions each of which is an ideal triangle. (See Figure 9.)

It is always possible to thicken up any leaves that support  $\delta$ -masses for the transverse measure to a parallel family of leaves with diffuse measure. Let us assume that we have performed this operation, so that our measured laminations have no  $\delta$ -masses.

Here is one theorem which relates actions on trees with measured laminations.

**Theorem 22** [16]. *Let  $M$  be a compact manifold, and let  $(\mathcal{L}, \mu)$  be a measured lamination in  $M$ . Suppose the following hold:*

- (1) *The leaves of the covering  $\tilde{\mathcal{L}}$  in the universal covering  $\tilde{M}$  of  $M$  are proper submanifolds.*
- (2) *If  $x, y \in \tilde{M}$ , then there is a path joining them which is transverse to  $\tilde{\mathcal{L}}$  and which meets each leaf of  $\tilde{\mathcal{L}}$  at most once.*

*Then there is a dual  $\mathbf{R}$ -tree  $T_{\mathcal{L}}$  to  $\tilde{\mathcal{L}}$  and an action of  $\pi_1(M)$  on  $T_{\mathcal{L}}$ . The branch points of  $T_{\mathcal{L}}$  correspond to the big complementary regions of  $\tilde{\mathcal{L}}$  in  $\tilde{M}$ , and thus there are only finitely many branch points modulo the action of  $\pi_1(M)$ . The regular points of  $T_{\mathcal{L}}$  correspond to the leaves of  $\mathcal{L}$  which are not boundary components of “big” complementary regions.*

*There is a continuous,  $\pi_1(M)$  equivariant map  $\widetilde{M} \rightarrow T_{\mathcal{L}}$  such that the inverse image of each branch point is a single complementary region and the preimage of each regular point is either a single leaf of the two leaves bounding a thin complementary component.*

In this way we see that “most” measured laminations give rise to dual actions of the fundamental group on  $\mathbf{R}$ -trees. There is also a partial converse to this result. Given an action of  $\pi_1(M)$  on an  $\mathbf{R}$ -tree  $T$  it is possible to construct a transverse,  $\pi_1(M)$ -equivariant map from the universal covering  $\widetilde{M}$  of  $M$  to  $T$ . Using this map, one can pull back a measured lamination from the tree and the metric on it. Thus, one produces a measured lamination associated to the action on the tree. This measured lamination dominates the original action. For example, if the lamination satisfies the hypothesis of Theorem 22, then dual to it there is another action of  $\pi_1(M)$  on an  $\mathbf{R}$ -tree. This new action maps in an equivariant manner to the original action. If the original action is minimal, then this map will be onto, and hence the new action dominates the original one in a precise sense.

There are several difficulties with this construction. First, it is not known that one can always do the construction so that the resulting lamination is dual to an action on a tree. Second, this construction is not unique; there are many choices of the transverse map. In many circumstances one hopes to find a “best” transverse map, but this can be hard to achieve.

Let  $M$  be a compact manifold. We say that an action of  $\pi_1(M)$  on an  $\mathbf{R}$ -tree is *geometric for  $M$*  if there is a measured lamination  $(\mathcal{L}, \mu)$  in  $M$  which satisfies the hypothesis of Theorem 22 and such that the action of  $\pi_1(M)$  dual to  $(\mathcal{L}, \mu)$  is isomorphic to the given action. We say that an action of an abstract finitely presented group  $G$  is *geometric* if there is a compact manifold  $M$  with  $\pi_1(M) = G$  and with the action geometric for  $M$ . One question arises: *Is every minimal action of a finitely presented group geometric?* If so it would follow that for a minimal action of a finitely presented group  $G$  on an  $\mathbf{R}$ -tree there are only finitely many branch points modulo the action of  $G$  and only finitely many directions from any point  $x$  of the tree modulo  $G_x$ . It is not known whether these statements are true in general.

In spite of these difficulties, measured laminations are an important tool for studying actions of finitely presented groups on trees. One extremely important result for measured laminations which gives a clue as to the dynamics of actions on  $\mathbf{R}$ -trees is the following decomposition result.

**Theorem 23** (cf. [13]). *Let  $(\mathcal{L}, \mu)$  be a measured lamination in a compact manifold  $M$ . Then there is a decomposition of  $\mathcal{L}$  into finitely many disjoint sublaminations each of which has support which is both open and closed in  $|\mathcal{L}|$ . Each of the sublaminations is of one of the following three types:*

- (i) *a parallel family of compact leaves,*
- (ii) *a twisted family of compact leaves with central member having nontrivial normal bundle in  $M$  and all the other leaves being two-sheeted sections of this normal bundle, and*
- (iii) *a lamination in which every leaf is dense.*

We call a lamination of the last type an *exceptional minimal lamination*.

One wonders if there is an analogous decomposition for actions of finitely presented groups on  $\mathbf{R}$ -trees.

### 5. APPLICATIONS AND QUESTIONS

In the analogy we have been developing between the theories of groups acting on simplicial trees and of groups acting on more general trees, there is one missing ingredient. There is no analogue for the combinatorial group theory in the simplicial case. There are some results that should be viewed as partial steps in this direction. In this section, we shall discuss some of these results and give applications of them to the spaces of hyperbolic structures on groups. Two things emerge, clearly, from this discussion. First of all, information about the combinatorial group theoretic consequences of the existence of an action of  $G$  on a  $\Lambda$ -tree has significant consequences for the space of hyperbolic structures on  $G$ . Second, most of the combinatorial group theoretic consequences to date have followed from the use of measured laminations and related ergodic considerations: first return maps, interval exchanges, etc.

**Applications to hyperbolic geometry.** Let us begin by formalizing the notion of a hyperbolic structure. Let  $G$  be a nonvirtually abelian, finitely generated group. For each  $n$  we denote by  $\mathcal{H}^n(G)$  the space of hyperbolic structures on  $G$ . By definition this means the conjugacy classes of discrete and faithful representations of  $G$  into the isometry group of hyperbolic  $n$ -space. Thought of another way a point in  $\mathcal{H}^n(G)$  is a complete Riemannian  $n$ -manifold with all sectional curvatures equal to  $-1$  and with fundamental group identified with  $G$ .

**Example 1.** Let  $G$  be the fundamental group of a closed surface of genus at least 2. Then  $\mathcal{H}^2(G)$  is the classical Teichmüller space. It is homeomorphic to a Euclidean space of dimension  $6g - 6$ .

**Example 2.** Let  $G$  be the fundamental group of a closed hyperbolic manifold of dimension  $n \geq 3$ . According to Mostow rigidity [18],  $\mathcal{H}^n(G)$  consists of a single point.

**Example 3.** Let  $G$  be the fundamental group of a closed hyperbolic manifold of dimension  $n$ . Then  $\mathcal{H}^{n+1}(G)$  can be of positive dimension (see [9]).

The material in §3 leads to the following result:

**Theorem 24** ([13, 11]). *There is a natural compactification of  $\mathcal{H}^n(G)$ . Each ideal point of this compactification is represented by a nontrivial action of  $G$  on  $\mathbf{R}$ -trees with property that the stabilizer of every nondegenerate segment of the tree is a virtually abelian subgroup of  $G$ .*

In the special case of surfaces this gives a compactification of the Teichmüller space of a surface of genus  $\geq 2$  by a space of actions of the fundamental group on  $\mathbf{R}$ -trees. It turns out that all these actions are geometric for the surface; i.e., the points at infinity in this compactification can be viewed as geodesic measured laminations. For more details see [22] and [13].

The Teichmüller space of a surface is acted on properly discontinuously by the mapping class group (the group of outer automorphisms of the fundamental group of the surface). The action of the mapping class group extends to

the compactification of Teichmüller. From this one can deduce various cohomological results for the mapping class group that make it look similar to an algebraic group. Motivated by this result, Culler-Vogtmann [5] have defined a space of free, properly discontinuous actions of a free group on  $\mathbf{R}$ -trees. This space is the analogue of the Teichmüller space for the outer automorphism group of the free group. The action of the outer automorphism group of the free group on this space extends to the compactification of this space of free, properly discontinuous actions inside the projective space of all actions of the free group on  $\mathbf{R}$ -trees. By [3] the limit points are actions of the free group on  $\mathbf{R}$ -trees with stabilizers of all nondegenerate segments being virtually abelian. Using this action, Culler-Vogtmann deduce many cohomological properties of the outer automorphism group of a free group.

Theorem 24 leads immediately to the question: *Which groups act nontrivially on  $\mathbf{R}$ -trees with the stabilizer of every nondegenerate segment in the tree being virtually abelian?* The answer for simplicial trees follows immediately from the analysis in §1.

**Proposition 25.** *A group  $G$  acts nontrivially on a simplicial tree with all edge stabilizers being virtually abelian if and only if  $G$  has either*

- (A) *a nontrivial free production with amalgamation with the amalgamating group being virtually abelian, or*
- (B) *an HNN-decomposition with the subgroup being virtually abelian.*

For fundamental groups of low dimensional manifolds, the answer is also known. This is a consequence of a study of measured laminations in 3-manifolds and, in particular, the ability to do surgery on measured laminations in 3-manifolds to make them incompressible.

**Theorem 26** ([14, 15]). *If  $M$  is a 3-manifold, then  $\pi_1(M)$  acts nontrivially on an  $\mathbf{R}$ -tree with the stabilizers of all nondegenerate segments being virtually abelian if and only if it has such an action on a simplicial tree.*

As an application we have

**Corollary 27.** *Let  $G$  be the fundamental group of a compact 3-manifold  $M$ . Then for all  $n$  the space  $\mathcal{H}^n(G)$  is compact unless  $G$  has a decomposition of type (A) or (B) in Proposition 25. In particular, if the interior of  $M$  has a complete hyperbolic structure and has incompressible boundary, then  $\mathcal{H}^n(G)$  is noncompact if and only if  $M$  has an essential annulus which is not parallel into  $\partial M$ .*

One suspects that the first statement is true without the assumption that  $G$  is a 3-manifold group.

*Conjecture.* Let  $G$  be a finitely presented group. Then for all  $n$  the space  $\mathcal{H}^n(G)$  is compact unless  $G$  has a decomposition of type (A) or (B) in Proposition 25.

One also suspects that any action of  $G$  on an  $\mathbf{R}$ -tree with segment stabilizers being virtually abelian can be approximated by an action of  $G$  on a simplicial

tree with the same property. This is known for surface groups by [22], but not in general.

The fundamental group of a closed hyperbolic  $k$ -manifold, for  $k \geq 3$ , has no decomposition of type (A) or (B). In line with the above suspicions we have a result proved by the geometric rather than algebraic means.

**Theorem 28** ([12]). *Suppose that  $G$  is the fundamental group of a closed hyperbolic manifold of dimension  $k \geq 3$ . Then for all  $n$ , the space  $\mathcal{H}^n(G)$  is compact.*

**Combinatorial group theory for groups acting on  $\mathbf{R}$ -trees.** Perhaps the simplest question about the combinatorial analogues of the simplicial case is the following: *which (finitely presented) groups act freely on  $\mathbf{R}$ -trees?* Another question of importance, suggested by the situation for valuations is: *Can one approximate a nontrivial action of a group  $G$  on an  $\mathbf{R}$ -tree by a nontrivial action of  $G$  on a simplicial tree? If the original action has stabilizers of all nondegenerate segments being virtually abelian, is there a simplicial approximation with this property? (For example, does the boundary of Culler-Vogtmann space have a dense subset consisting of simplicial actions?)*

We finish by indicating some of the partial results concerning these two questions. Because of the existence of geodesic laminations with simply connected complementary regions, surface groups act freely on  $\mathbf{R}$ -trees (see [16]). Of course, any subgroup of  $\mathbf{R}$  also acts freely on an  $\mathbf{R}$ -tree. It follows easily that any free product of these groups acts freely on an  $\mathbf{R}$ -tree. The question is whether there are any other groups which act freely. By Theorem 26, for 3-manifold groups the answer is no.

We say that a group is *indecomposable* if it is not a nontrivial free product. Clearly, it suffices to classify indecomposable groups which act freely on  $\mathbf{R}$ -trees. In [10] and [17] the class of finitely presented, indecomposable groups which have a nontrivial decomposition as a free product with amalgamation or HNN-decomposition, where in each case the subgroup is required to be virtually abelian, were studied. It was shown that if such a group acts freely on an  $\mathbf{R}$ -tree, then it is either a surface group or a free abelian group. This result is proved by invoking the theory of measured laminations and results from ergodic theory. Another result which follows from using the same techniques is: A minimal free action of an indecomposable, noncyclic group  $G$  on an  $\mathbf{R}$ -tree  $T$  is mixing in the following sense. Given any two nondegenerate segments  $I$  and  $J$  in  $T$ , there is a decomposition  $I = I_1 \cup \dots \cup I_p$  and group elements  $g_i \in G$  such that for all  $i$  we have  $g_i \cdot I_i \subset J$ . In particular, the orbit  $G \cdot J$  of  $J$  is the entire tree.

In a different direction, in [6] Gillet-Shalen proved that if  $G$  acts freely on an  $\mathbf{R}$ -tree which is induced by base change from a  $\Lambda$ -tree where  $\Lambda$  is a subgroup of  $\mathbf{R}$  which generates a rational vector space of rank at most 2, then  $G$  is a free product of surface groups and free abelian groups. By the same techniques they, together with Skora [7], show that actions on such  $\Lambda$ -trees can be approximated by actions on simplicial trees (preserving the virtually abelian segment stabilizer condition of relevant).

Recently, Rips has claimed that the only finitely generated groups which act freely on  $\mathbf{R}$ -trees are free products of surface groups and free abelian groups.

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