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SOME NON-ANALYTIC-HYPOELLIPTIC SUMS OF SQUARES OF VECTOR FIELDS

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ABSTRACT. Certain second-order partial differential operators, which are expressed as sums of squares of real-analytic vector fields in $\mathbb{R}^3$ and which are well known to be C^∞ hypoelliptic, fail to be analytic hypoelliptic.

1. INTRODUCTION

A differential operator L is said to be analytic hypoelliptic if whenever u is a distribution such that Lu is real-analytic in some open set U, then u is necessarily also real-analytic in U. Elliptic operators with analytic coefficients are analytic hypoelliptic, as are certain classes of subelliptic operators [GS, M2, S, Ta, Tp, Tv1, Tv2]. It has been known for some time that many subelliptic operators—whose solutions are necessarily C^∞—nonetheless fail to be analytic hypoelliptic; among the examples now known are [BG, M1, He, PR, HH, CG]. A substantial no-man's-land persists, in which neither alternative has been proved, even in rather simple cases. In this note are announced negative results for certain second-order operators. We hope that these will serve as models for larger classes of operators, rather than being mere isolated examples.

In $\mathbb{R}^3$ with coordinates x, y, t set

$$(0) \qquad X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} - mx^{m-1}\frac{\partial}{\partial t},$$

and

$$L = X^2 + Y^2,$$

where $m \geq 2$ is an integer. Then L is hypoelliptic in the C^∞ sense [H1, K]; when $m = 2$, it is analytic hypoelliptic [M2, Ta, Tv2]. For $m \geq 3$ an odd integer, however, it is not analytic hypoelliptic. This was proved for $m = 3$ in [He, PR], and extended to larger m in [HH], but by a method which does not apply for m even. In [CG] it was found that $\bar{\partial}_b \circ \bar{\partial}_b^*$ fails to be microlocally analytic hypoelliptic in the appropriate part of phase space, on the CR manifold $\{\Im(z_2) = [\Re(z_1)]^m\}$, where $m \geq 4$ is even. In appropriate coordinates for this manifold, $-\bar{\partial}_b \circ \bar{\partial}_b^* = (X + iY) \circ (X - iY) = X^2 + Y^2 - i[X, Y]$, where X, Y are as in (0).

Theorem 1. *For any even integer $m \geq 4$, L is not analytic hypoelliptic.*

Despite the similarity to results just cited, this does not follow from previous methods. The proof is rooted in a phenomenon discovered for $\bar{\partial}_b \circ \bar{\partial}_b^*$ in [CG],

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but that argument relied heavily upon an explicit formula for the Szegő kernel [N], for which there appears to be no analogue in the present situation.

To place Theorem 1 in context, consider two real vector fields X, Y in $\mathbb{R}^3$ with analytic coefficients, and suppose them to be linearly independent at each point. Say that a point $a \in \mathbb{R}^3$ is of type 2 if $X, Y, [X, Y]$ span the tangent space to $\mathbb{R}^3$ at a. A general result [Tv2, Ta, M2] guarantees analytic hypoellipticity at any point of type 2, leaving open the question of what sort of degeneracy is permitted. The following conjecture has been suggested in a more general form by Trèves [Tv2]: $L = X^2 + Y^2$ fails to be analytic hypoelliptic at a if in any neighborhood of a there exists a real curve γ, with $\gamma'(0) \neq 0$, such that

- $\gamma(t)$ is not a point of type 2 for any $|t| < \varepsilon$, and
- $\gamma'(t)$ belongs to the span of $X(\gamma(t))$, $Y(\gamma(t))$ for every $|t| < \varepsilon$.

One may hope that analytic hypoellipticity holds in all other cases. In the special case of Theorem 1, the plane $x = 0$ is foliated by a one-parameter family of such curves γ.

More recently we have built on the analysis outlined below to prove that analytic hypoellipticity breaks down for $X^2 + Y^2$, with $X = \partial_x$ and $Y = \partial_y - b'(x)\partial_t$, whenever b vanishes to order exactly m at some point, with $m \in \{3, 4, 5, \dots\}$.

2. Outline of proof

Let ζ, τ be variables dual to y, t. Taking a partial Fourier transform in these variables reduces the analysis of L to that of a two-parameter family of ordinary differential operators:

$$-\frac{d^2}{dx^2} + (\zeta - \tau m x^{m-1})^2.$$

A simple change of variables reduces the general case $\tau \neq 0$ to $\tau = 1$, so we set

$$\mathscr{L}_\zeta = -\frac{d^2}{dx^2} + (\zeta - m x^{m-1})^2.$$

It is well known [H2] that in order to prove that L is not analytic hypoelliptic, it suffices to demonstrate the next result (which is already known [PR, HH] for odd m).

Theorem 2. *Let $m \geq 3$ be an integer. Then there exist $\zeta \in \mathbb{C}$ and $f \in L^\infty(\mathbb{R})$, not identically equal to zero, satisfying $\mathscr{L}_\zeta f \equiv 0$.*

For then, assuming that ζ has strictly positive imaginary part, one may set

$$F(x, y, t) = \int_1^\infty e^{i\tau t + i\tau^{1/m}\zeta y} f(\tau^{1/m} x)\, d\tau$$

in the region $y > 0$. Then $F \in C^\infty$, and $LF \equiv 0$. If $f(0) \neq 0$, one calculates readily, via a change of the contour of integration, that

$$\left| \frac{\partial^k}{\partial t^k} F(0, 1, 0) \right| \geq \delta^{k+1} (mk)!$$

for some $\delta > 0$. Thus F is not real-analytic. If $f(0)$ does vanish, then $\frac{d}{dx} f(0) \neq 0$ and essentially the same reasoning applies to $\frac{\partial}{\partial x}\frac{\partial^k}{\partial t^k} F$. It is easy to check that $\mathscr{L}_\zeta$ has a strictly positive lowest eigenvalue for each $\zeta \in \mathbb{R}$, and that for any ζ satisfying the conclusion of Theorem 2, $\bar{\zeta}$ does also; so the assumption above is legitimate.

We have only an indirect proof of the existence of (infinitely many) ζ with the property desired. Set $\gamma = -(m-1)/2$, and $\Phi_\zeta(x) = \zeta x - x^m$. Since the coefficient of the first-order part of $\mathscr{L}_\zeta$ is zero, the Wronskian of any two solutions of $\mathscr{L}_\zeta$ is a constant function of x. In the next lemma we will have two such solutions for each ζ, so their Wronskian will be a function of ζ alone.

Lemma 3. *Let $m \geq 4$ be an even integer. For each $\zeta \in \mathbb{C}$ there exist functions f_ζ^+ and f_ζ^- defined on $\mathbb{R}$ which satisfy $\mathscr{L}_\zeta f_\zeta^\pm \equiv 0$ and*

$$(1) \qquad \left| f_\zeta^\pm(x) - e^{\Phi_\zeta(x)}|x|^\gamma \right| = O(|e^{\Phi_\zeta(x)}| \cdot |x|^{\gamma-1}) \quad \text{as } x \to \pm\infty,$$

respectively. These functions are unique, and depend holomorphically on ζ. Their Wronskian, W, satisfies

$$(2) \qquad |W(\zeta)| \leq C \exp(C|\zeta|^{m/(m-1)}) \quad \forall \zeta \in \mathbb{C}$$

for some finite C and

$$(3) \qquad |W(\zeta)| \geq \delta \exp(\delta|\zeta|^{m/(m-1)}) \quad \forall \zeta \in \mathbb{R},$$

for some $\delta > 0$.

Now, W must have at least one zero. If not, then the real part of $\log W$ would be a harmonic function on $\mathbb{C}^1$ with polynomial growth at infinity, hence would be a polynomial. By (2) and (3), its degree would have to be $m/(m-1)$. But for $m \geq 3$, $m/(m-1)$ is not an integer.[1]

If $W(\zeta) = 0$, then f_ζ^- is a constant multiple of f_ζ^+. Hence both decay exponentially as $x \to \pm\infty$, therefore certainly remain bounded. Thus f_ζ^+ is the function sought.

The same reasoning can be made to apply for odd $m \geq 3$, with a suitable modification of (1). Further argument shows that for any $\alpha \in \mathbb{R}$, the operator $X^2 + Y^2 + i\alpha[X, Y]$ fails to be analytic hypoelliptic. Related results appear in [C1, C2, C4].

The proof of Lemma 3 is entirely elementary; details will appear elsewhere [C3]. The existence of solutions $f_\zeta^\pm$ with the prescribed asymptotics is a special case of a standard result in the theory of ordinary differential equations with irregular singular points at infinity [CL].

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