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# THE $A_\ell$ AND $C_\ell$ BAILEY TRANSFORM AND LEMMA

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ABSTRACT. We announce a higher-dimensional generalization of the Bailey Transform, Bailey Lemma, and iterative "Bailey chain" concept in the setting of basic hypergeometric series very well-poised on unitary $A_\ell$ or symplectic $C_\ell$ groups. The classical case, corresponding to $A_1$ or equivalently $\mathrm{U}(2)$, contains an immense amount of the theory and application of one-variable basic hypergeometric series, including elegant proofs of the Rogers-Ramanujan-Schur identities. In particular, our program extends much of the classical work of Rogers, Bailey, Slater, Andrews, and Bressoud.

## 1. INTRODUCTION

The purpose of this paper is to announce a higher-dimensional generalization of the Bailey Transform [2] and Bailey Lemma [2] in the setting of basic hypergeometric series very well-poised on unitary [19] or symplectic [14] groups. Both types of series are directly related [14, 18] to the corresponding Macdonald identities. The series in [19] were strongly motivated by certain applications of mathematical physics and the unitary groups $\mathrm{U}(n)$ in [10, 11, 15, 16]. The unitary series use the notation $A_\ell$, or equivalently $\mathrm{U}(\ell+1)$; the symplectic case, $C_\ell$. The classical Bailey Transform, Lemma, and very well-poised basic hypergeometric series correspond to the case $A_1$, or equivalently $\mathrm{U}(2)$.

The classical Bailey Transform and Bailey Lemma contain an immense amount of the theory and application of one-variable basic hypergeometric series [2, 12, 25]. They were ultimately inspired by Rogers' [24] second proof of the Rogers-Ramanujan-Schur identities [23]. The Bailey Transform was first formulated by Bailey [8], utilized by Slater in [25], and then recast by Andrews [4] as a fundamental matrix inversion result. This last version of the Bailey Transform has immediate applications to connection coefficient theory and "dual" pairs of identities [4], and $q$-Lagrange inversion and quadratic transformations [13].

The most important application of the Bailey Transform is the Bailey Lemma. This result was mentioned by Bailey [8; §4], and he described how the proof would work. However, he never wrote the result down explicitly and thus missed the full power of *iterating* it. Andrews first established the Bailey Lemma explicitly in [5] and realized its numerous possible applications in terms of the iterative "Bailey chain" concept. This iteration mechanism enabled him to derive many $q$-series identities by "reducing" them to more elementary ones. For example,

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the Rogers-Ramanujan-Schur identities can be reduced to the $q$-binomial theorem. Furthermore, general multiple series Rogers-Ramanujan-Schur identities are a direct consequence of iterating suitable special cases of Bailey's Lemma. In addition, Andrews notes that Watson's $q$-analog of Whipple's transformation is an immediate consequence of the second iteration of one of the simplest cases of Bailey's Lemma. Continued iteration of this same case yields Andrews' [3] infinite family of extensions of Watson's $q$-Whipple transformation. Even Whipple's original work [26, 27] fits into the $q = 1$ case of this analysis. Paule [22] independently discovered important special cases of Bailey's Lemma and how they could be iterated. Essentially all the depth of the Rogers-Ramanujan-Schur identities and their iterations is embedded in Bailey's Lemma.

The process of iterating Bailey's Lemma has led to a wide range of applications in additive number theory, combinatorics, special functions, and mathematical physics. For example, see [2, 5, 6, 7, 9].

The Bailey Transform is a consequence of the terminating ${}_4\phi_3$ summation theorem. The Bailey Lemma is derived in [1] directly from the ${}_6\phi_5$ summation and the matrix inversion formulation [4, 13] of the Bailey Transform. We employ a similar method in the $A_\ell$ and $C_\ell$ cases by starting with a suitable, higher-dimensional, terminating ${}_6\phi_5$ summation theorem extracted from [19] and [14], respectively. The $A_\ell$ proofs appear in [20, 21], and the $C_\ell$ case is established in [17]. Many other consequences of the $A_\ell$ and $C_\ell$ generalizations of Bailey's Transform and Lemma will appear in future papers. These include $A_\ell$ and $C_\ell$ $q$-Pfaff-Saalschütz summation theorems, $q$-Whipple transformations, connection coefficient results, and applications of iterating the $A_\ell$ or $C_\ell$ Bailey Lemma.

## 2. Results

Throughout this article, let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{N}$, and $\boldsymbol{y}$ be vectors of length $\ell$ with nonnegative integer components. Let $q$ be a complex number such that $|q| < 1$. Define

$$(2.1a)\qquad (\alpha)_\infty \equiv (\alpha;\ q)_\infty := \prod_{k\geq 0}(1-\alpha q^k)$$

and, thus,

$$(2.1b)\qquad (\alpha)_n \equiv (\alpha;\ q)_n := (\alpha)_\infty \,/\, (\alpha q^n)_\infty .$$

Define the Bailey transform matricies, $M$ and $M^*$, as follows.

**Definition** ($\boldsymbol{M}$ and $\boldsymbol{M}^*$ for $\boldsymbol{A_\ell}$). Let $a, x_1, \dots, x_\ell$ be indeterminate. Suppose that none of the denominators in (2.2a–b) vanishes. Then let

$$(2.2a)\qquad M(\boldsymbol{i};\boldsymbol{j};A_\ell) := \prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)_{i_r-j_r}^{-1}\ \prod_{k=1}^{\ell}\left(aq\frac{x_k}{x_\ell}\right)_{i_k+(j_1+\cdots+j_\ell)}^{-1};$$

and

(2.2b)

$$M^*(\boldsymbol{i};\ \boldsymbol{j};\ A_\ell) := \prod_{k=1}^{\ell}\left[1 - a\frac{x_k}{x_\ell}q^{i_k+(i_1+\cdots+i_\ell)}\right]\prod_{k=1}^{\ell}\left(aq\frac{x_k}{x_\ell}\right)_{j_k+(i_1+\cdots+i_\ell)-1} \times \prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r} (-1)^{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}\, q^{\binom{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}{2}}.$$

**Definition** ( $M$ and $M^*$ for $C_\ell$ ). Let $x_1, \ldots, x_\ell$ be indeterminate. Suppose that none of the denominators in (2.3a–b) vanishes. Then let

$$M(\boldsymbol{i};\boldsymbol{j};C_\ell) := \prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\left(qx_rx_sq^{j_r+j_s}\right)^{-1}_{i_r-j_r}\right]; \tag{2.3a}$$

and

(2.3*b*)

$$M^*(\boldsymbol{i};\boldsymbol{j};C_\ell) := \prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\left(x_rx_sq^{j_r+i_s}\right)^{-1}_{i_r-j_r}\right]\prod_{1\le r<s\le \ell}\left[\frac{1-x_rx_sq^{j_r+j_s}}{1-x_rx_sq^{i_r+i_s}}\right] \times (-1)^{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}q^{\binom{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}{2}}.$$

As in the classical case [1], we have the following theorem.

**Theorem** (Bailey Transform for $A_\ell$ and $C_\ell$ ). *Let $G = A_\ell$ or $C_\ell$. Let $M$ and $M^*$ be defined as in* (2.2) *and* (2.3), *with rows and columns ordered lexicographically. Then $M$ and $M^*$ are inverse, infinite, lower-triangular matricies. That is,*

$$\prod_{k=1}^{\ell}\delta(i_k, j_k) = \sum_{\substack{j_k\le y_k\le i_k\\ k=1,2,\ldots,\ell}} M(\boldsymbol{i};\ \boldsymbol{y};\ G)\, M^*(\boldsymbol{y};\ \boldsymbol{j};\ G), \tag{2.4}$$

*where $\delta(r, s) = 1$ if $r = s$, and $0$ otherwise.*

Equations (2.2) and (2.3) motivate the definition of the $A_\ell$ and $C_\ell$ Bailey pair.

**Definition** ( $G$-Bailey Pair). Let $G = A_\ell$ or $C_\ell$. Let $N_k \ge 0$ be integers for $k = 1, 2, \ldots, \ell$. Let $A = \{A_{(\boldsymbol{y};G)}\}$ and $B = \{B_{(\boldsymbol{y};\ G)}\}$ be sequences. Let $M$ and $M^*$ be as above. Then we say that $A$ and $B$ form a $G$-Bailey Pair if

$$B_{(\boldsymbol{N};G)} = \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}} M(\boldsymbol{N};\boldsymbol{y};G)\, A_{(\boldsymbol{y};G)}. \tag{2.5}$$

As a consequence of the Bailey transform, (2.4), and the definition of the $G$-Bailey pair, (2.5), we have the following result.

**Corollary** (Bailey Pair Inversion). *$A$ and $B$ satisfy equation* (2.5) *if and only if*

(2.6)
$$A_{(\boldsymbol{N};G)} = \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\dots,\ell}} M^*(\boldsymbol{N};\boldsymbol{y};G)\, B_{(\boldsymbol{y};G)}.$$

Define the sequences $A' = \{A'_{(\boldsymbol{y};A_\ell)}\}$ and $B' = \{B'_{(\boldsymbol{y};A_\ell)}\}$ by

(2.7a)
$$A'_{(\boldsymbol{N};A_\ell)} := \prod_{k=1}^{\ell}\left(\frac{aq}{\rho}\frac{x_k}{x_\ell}\right)^{-1}_{N_k} \prod_{k=1}^{\ell}\left(\sigma\frac{x_k}{x_\ell}\right)_{N_k} \times \frac{(\rho)_{N_1+\cdots+N_\ell}}{(aq/\sigma)_{N_1+\cdots+N_\ell}}(aq/\rho\sigma)^{N_1+\cdots+N_\ell}\, A_{(\boldsymbol{N};\,A_\ell)}$$

and

(2.7b)
$$B'_{(\boldsymbol{N};A_\ell)} := \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\dots,\ell}} \left\{\prod_{k=1}^{\ell}\left[\left(\sigma\frac{x_k}{x_\ell}\right)_{y_k}\left(\frac{aq}{\rho}\frac{x_k}{x_\ell}\right)^{-1}_{N_k}\right]\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{y_r-y_s}\right)^{-1}_{N_r-y_r} \times \frac{(aq/\rho\sigma)_{(N_1+\cdots+N_\ell)-(y_1+\cdots+y_\ell)}\,(\rho)_{y_1+\cdots+y_\ell}}{(aq/\sigma)_{N_1+\cdots+N_\ell}} \times (aq/\rho\sigma)^{y_1+\cdots+y_\ell}\, B_{(\boldsymbol{y};\,A_\ell)}\right\}$$

Define the sequences $A' = \{A'_{(\boldsymbol{y};C_\ell)}\}$ and $B' = \{B'_{(\boldsymbol{y};C_\ell)}\}$ by

(2.8a)
$$A'_{(\boldsymbol{N};C_\ell)} := \prod_{k=1}^{\ell}\left[\frac{(\alpha x_k)_{N_k}\,(qx_k\beta^{-1})_{N_k}}{(\beta x_k)_{N_k}\,(qx_k\alpha^{-1})_{N_k}}\right]\left(\frac{\beta}{\alpha}\right)^{N_1+\cdots+N_\ell} A_{(\boldsymbol{N};C_\ell)}$$

and

(2.8b)
$$B'_{(\boldsymbol{N};\,C_\ell)} := \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\dots,\ell}} \left\{cr\prod_{k=1}^{\ell}\left[\frac{(\alpha x_k)_{y_k}\,(qx_k\beta^{-1})_{y_k}}{(\beta x_k)_{N_k}\,(qx_k\alpha^{-1})_{N_k}}\right]\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{y_r-y_s}\right)^{-1}_{N_r-y_r} \times \prod_{1\le r<s\le\ell}\left[(qx_rx_sq^{y_r+y_s})^{-1}_{N_s-y_s}\,(qx_rx_sq^{N_s-y_s})^{-1}_{N_r-y_r}\right] \times \left(\frac{\beta}{\alpha}\right)_{(N_1+\cdots+N_\ell)-(y_1+\cdots+y_\ell)}\left(\frac{\beta}{\alpha}\right)^{y_1+\cdots+y_\ell} B_{(\boldsymbol{y};\,C_\ell)}\right\}$$

These definitions lead to our generalization of Bailey's lemma.

**Theorem** (The $G$-generalization of Bailey's Lemma). *Let $G = A_\ell$ or $C_\ell$. Suppose $A = \{A_{(\boldsymbol{N};G)}\}$ and $B = \{B_{(\boldsymbol{N};G)}\}$ form a G-Bailey Pair. If $A' = \{A'_{(\boldsymbol{N};G)}\}$ and $B' = \{B'_{(\boldsymbol{N};G)}\}$ are as above, then $A'$ and $B'$ also form a G-Bailey Pair.*

## 3. Sketches of Proofs

*Proof of* (2.4). In each case, $A_\ell$ and $C_\ell$, we begin with a terminating ${}_4\phi_3$ summation theorem. In the $C_\ell$ case, it is first necessary to specialize Gustafson's $C_\ell$ ${}_6\psi_6$ summation theorem, see [14], terminate it from below and then from above, and further specialize the resulting terminating ${}_6\phi_5$ to yield a terminating ${}_4\phi_3$. In both the $A_\ell$ and $C_\ell$ cases, the ${}_4\phi_3$ is modified by multiplying both the sum and product sides by some additional factors. Finally, that result is transformed term-by-term to yield the sum side of (2.4). □

*Proof of* (2.6). Equation (2.6) follows directly from the definition, (2.5), and the termwise nature of the calculations in the proof of (2.4). □

*Proof of Bailey's Lemma.* The definitions in (2.7) and (2.8) are substituted into (2.5). After an interchange of summation, the inner sum is seen to be a special case of the appropriate ${}_6\phi_5$. The ${}_6\phi_5$ is then summed, and the desired result follows. □

Detailed proofs of the $C_\ell$ case will appear in [17], as will a discussion of the $C_\ell$ Bailey chain and a connection coefficient result associated with the $C_\ell$ Bailey Transform.

## References


1. A. K. Agarwal, G. Andrews, and D. Bressoud, *The Bailey lattice*, J. Indian Math. Soc. **51** (1987), 57–73.
2. G. E. Andrews, *q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra,* CBMS Regional Con. Ser. in Math., no. 66, Conf. Board Math. Sci., Washington, DC, 1986.
3. ——, *Problems and prospects for basic hypergeometric functions*, Theory and Applications of Special Functions (R. Askey, ed.), Academic Press, New York, 1975, pp. 191–224.
4. ——, *Connection coefficient problems and partitions*, Proc. Sympos. Pure Math., (D. Ray-Chaudhuri, ed.), vol. 34, Amer. Math. Soc., Providence, RI, 1979, pp. 1–24.
5. ——, *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. **114** (1984), 267–283.
6. G. E. Andrews, R. J. Baxter, and P. J. Forrester, *Eight-vertex SOS model and generalized Rogers-Ramanujan-type identities*, J. Statist. Phys. **35** (1984), 193–266.
7. G. E. Andrews, F. J. Dyson, and D. Hickerson, *Partitions and indefinite quadratic forms*, Invent. Math. **91** (1988), 391–407.
8. W. N. Bailey, *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **50** (1949), 1–10.
9. R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London and New York, 1982.
10. L. C. Biedenharn and J. D. Louck, *Angular momentum in quantum physics: Theory and applications*, Encyclopedia of Mathematics and Its Applications, (G.-C. Rota, ed.), vol. 8, Addison-Wesley, Reading, MA, 1981.
11. ——, *The Racah-Winger algebra in quantum theory*, Encyclopedia of Mathematics and Its Applications, (G.-C. Rota, ed.), vol. 9, Addison-Wesley, Reading, MA, 1981.
12. G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and Its Applications, (G.-C. Rota, ed.), vol. 35, Cambridge University Press, Cambridge, 1990.

13. I. Gessel and D. Stanton, *Applications of q-Lagrange inversion to basic hypergeometric series*, Trans. Amer. Math. Soc. **277** (1983), 173–201.

14. R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hypergeometric series very well-poised on semi-simple Lie algebras*, Ramanujan International Symposium on Analysis (December 26th to 28th, 1987, Pune, India) (N. K. Thakare, ed.), 1989, pp. 187–224.

15. W. J. Holman, III, *Summation Theorems for hypergeometric series in* $\mathrm{U}(n)$, SIAM J. Math. Anal. **11** (1980), 523–532.

16. W. J. Holman III, L. C. Biedenharn, and J. D. Louck, *On hypergeometric series well-poised in* $\mathrm{SU}(n)$, SIAM J. Math. Anal. **7** (1976), 529–541.

17. G. M. Lilly and S. C. Milne, *The* $C_\ell$ *Bailey transform and Bailey lemma*, preprint.

18. S. C. Milne, *An elementary proof of the Macdonald identities for* $A_\ell^{(1)}$, Adv. in Math. **57** (1985), 34–70.

19. ——, *Basic hypergeometric series very well-poised in* $\mathrm{U}(n)$, J. Math. Anal. Appl. **122** (1987), 223–256.

20. ——, *Balanced* ${}_3\phi_2$ *summation theorems for* $\mathrm{U}(n)$ *basic hypergeometric series*, (in preparation).

21. ——, *A* $\mathrm{U}(n)$ *generalization of Bailey's lemma*, in preparation.

22. P. Paule, *Zwei neue Transformationen als elementare Anwendungen der q-Vandermonde Formel*, Ph.D. thesis, 1982, University of Vienna.

23. L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

24. ——, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc (2) **16** (1917), 315–336.

25. L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, London and New York, 1966.

26. F. J. W. Whipple, *On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum*, Proc. London Math. Soc. (2) **24** (1924), 247–263.

27. ——, *Well-poised series and other generalized hypergeometric series*, Proc. London Math. Soc. (2) **25** (1926), 525–544.

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