BOOK REVIEWS

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Spectral theory of families of self-adjoint operators, by Yu. S. Samoilenko, Kluwer Academic Publishers, 1991, 293 pp., \$129.00, ISBN 0-7923-0703-8

John von Neumann wrote in [vN1], "As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality," it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art.* This need not be bad if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and the discipline will become a disorganized mass of details and complexities." The study of operators has run that "grave danger" in the past. Von Neumann et al. studied operators as a possible mathematical treatment of quantum mechanics. At the present time there are but a handful of books dealing with the study of operators in this manner (a noteworthy exception being [RS1]). Most books give priority to the study of bounded operators over that of unbounded operators, and selfadjoint operators are sometimes considered the "well-known case." A few monographs have taken the approach of studying the mathematics of quantum mechanics without referring to (or requiring knowledge of) the physics of quantum mechanics (see for example [BS1]). The book under review fits into this last category and fills an important gap on the subject.

The key aspect of operator theory that concerns quantum mechanics is called spectral theory. The book under review quotes [RS1] as saying, "The spectral theorem together with the multiplicity theory is one of the pearls of mathematics." Spectral theory can be described as a plan of action for the study of operators, outlined as follows:

(a) Study the simplest operators. Multiplication by a real number is the simplest selfadjoint operator on a separable Hilbert space.

(b) Introduce topological and measure structures on the spectrum.

(c) Prove the "Spectral Theorem" that, in its one variable version, says that any operator is (unitarily equivalent) to an operator formed by a "combination" of the simplest operators.

After succeeding in developing (a), (b), and (c), we can

(d) Describe the unitary invariants.
(e) Construct a functional calculus.
(f) Study specific operators.

Most books about spectral theory follow the plan (or parts of it) outlined above for a single (normal or selfadjoint) bounded operator (see [RN1] for a classical treatment, [H1] for a quick and clean approach to the Spectral Theorem and the theory of spectral multiplicity, and [BS1] for a somewhat more complete and up-to-date treatment). Some include an exposition on unbounded selfadjoint operators (see [C1]). Others (like [BS1]) go a little farther by studying finite families of selfadjoint operators. Samoilenko's book studies infinite families of selfadjoint operators subject to some type of algebraic condition (e.g., commutativity, anticommutativity, or a Lie relation). Such families are found in the study of physical systems with infinite degrees of freedom in quantum physics, statistical mechanics, and field theory.

Developing a spectral theory for more than one selfadjoint operator without any further condition constitutes an extremely hard problem. Even for pairs of operators very little is known in this generality (cf. [E1]).

The concepts of joint spectrum and joint resolution of identity can successfully be used to develop a spectral theory for a countable family of commuting selfadjoint operators (CSO). Let $\mathscr{H}$ be a separable complex Hilbert space and $\{A_j\}_{j=1}^{\infty}$ be a countable collection of selfadjoint operators. Furthermore, assume A_j commutes with A_k for any $j, k = 1, 2, \ldots$. For a subfamily $A_1, A_2, \ldots, A_n$ a joint n-dimensional resolution of the identity can be defined as an operator-valued measure $E(\cdot)$ defined on the σ-algebra of Borel sets of $\mathbb{R}^n$, with the following properties:

(1)
$$A_j = \int_{\mathbb{R}^n} \lambda_j \, dE(\lambda_1, \lambda_2, \ldots, \lambda_n),$$
where the integral converges in the strong sense.

(2) The operator $E(\Delta)$ is a projection for any Borel subset Δ of $\mathbb{R}^n$. Also, $E(\varnothing) = 0$ and $E(\mathbb{R}^n) = I$.

(3) The measure $E(\cdot)$ is σ-additive (i.e.,
$$E\left(\bigcup_{j=1}^{\infty} \Delta_j\right) = \sum_{j=1}^{\infty} E(\Delta_j), \qquad \Delta_j \cap \Delta_k = \varnothing \text{ for all } j \neq k).$$

(4) The measure $E(\cdot)$ is orthogonal (i.e., $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2))$, for Borel subset Δ_1, Δ_2 of $\mathbb{R}^n$.

If Δ is a cartesian product of intervals, $E(\Delta)$ is defined as the product of the measures of those intervals. The measure can be extended to Borel sets in the usual way.

We can now use these measures to define an operator-valued measure on $\mathbb{R}^\infty$ (the cartesian product of countably many copies of $\mathbb{R}$) by considering the infinite product measure $\bigotimes_{j=1}^{\infty} dE_j(\lambda_j)$. The construction of this measure is the proof of the Spectral Theorem for a countable family of CSO. The joint spectrum of $\{A_j\}_{j=1}^{\infty}$ can be defined as the intersection of all closed subsets of $\mathbb{R}^\infty$ with full operator-valued $\bigotimes_{j=1}^{\infty} dE_j(\lambda_j)$-measure. This theory imitates the development of spectral theory for a single selfadjoint operator. Samoilenko's book presents a good self-contained version of this theory, including an interesting chapter (Chapter 3) on applications to differential operators acting on spaces of functions of infinitely (countably) many variables.

A similar theory can be developed for a countable family $\{A_j\}_{j=1}^{\infty}$ of selfadjoint operators on $\mathscr{H}$ that anticommute (i.e., $A_jA_k = -A_kA_j$ for all $j, k = 1, 2, \ldots$). More general relations (e.g., $A_jA_k + A_kA_j = \delta_{jk}$) or the inclusion of uncountable families (both of which occur in the study of quantum mechanics) produce problems beyond the reach of current techniques.

The study of CSO families is connected with the study of random sequences and with the theory of unitary representations of inductive limits of commutative, connected, simply connected Lie groups. We can consider a subcollection $\{A_j\}_{j=1}^{n}$ as a representation of the basis of the real (n-dimensional) Lie algebra. Therefore, we can study properties of the family $\{A_j\}_{j=1}^{\infty}$ by studying representations of inductive limits of certain Lie groups. By dropping the commutativity condition we can study families with more general (Lie) algebraic relations.

A very important technique in this field consists of using C^*-algebras and their representations. Also, by considering the family $\{A_j\}_{j=1}^{\infty}$ of selfadjoint operators as a nonrealized random field, we can use the theory of (noncommutative) random sequences to get information about the family. The last part of Samoilenko's book presents a brief introduction of these topics.

I believe that the book will be of interest to any expert in quantum mechanics and operator theory. I can find only one serious drawback to the book, more a fault of the subject than the author! Its logical prerequisites ("a basic university course including an exposition of the theory of unbounded selfadjoint operators") are far away from the real prerequisites necessary to comprehend this book. They include:

(1) "... some basic knowledge of measure theory on sequence spaces."

(2) "... the fundamentals of the theory of unitary representations of Lie groups and Lie algebras."

(3) "... the fundamentals of the theory of $*$-algebras and their representations."

Achieving knowledge of all these diverse subjects is not an easy task. Nevertheless, anyone who can fulfill this wide range of prerequisites will certainly be rewarded upon reading this book.

References

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Lagrangian manifolds and the Maslov operator, by A. S. Mishchenko, V. E. Shatalov, and B. Yu. Sternin. (Translated from the Russian by Dana Mackenzie.) Springer-Verlag, Berlin, Heidelberg, New York, 1990, 395 pp., $69.50. ISBN 0-540-13613-4

From the foreword: "This book presents Maslov's canonical operator method for finding asymptotic solutions of pseudodifferential equations." Nothing in the foreword about Hörmander's theory of Fourier integral operators, not even Hörmander's name is mentioned in what the authors call "a more or less full list of works in which Maslov's theory is being developed or used." On the other hand, the book contains an appendix on Fourier-Maslov Integral Operators, in the introduction of which it is stated: "Moreover, it was later clarified that Maslov's canonical operator method, when applied in the situation of Fourier integral operators, precisely coincided with the latter."

If the subjects are the same, then it seems rather pointless to publish a translation of the Russian book written in 1978, in the presence of the very complete and clear exposition of the theory of Fourier integral operators in Volume 4 of Hörmander [H14]. If the subjects are not identical, then it is worthwhile to also have a presentation in English of Maslov's canonical operator method. In concordance with the authors of the book, I think that both assumptions hold.

In order to explain this, I would like to begin with a short description of the objects of study. Having these available, it will be better possible to compare the various theories and to describe the contents of the book in more detail.

High frequency waves are functions of the form

$$u(x, \omega) = e^{i\omega\phi(x)} a(x, \omega). \tag{1}$$

Here ω is a positive real number representing the *frequency*, and one is interested in asymptotic formulas as $\omega \to \infty$. The *phase function* $\phi(x)$ is a smooth real-valued function of the position variables $x = (x_1, \dots, x_n)$. The *amplitude* $a(x, \omega)$ is a complex-valued smooth function of x that has an asymptotic expansion in decreasing powers of ω of the form

$$a(x, \omega) \sim \sum_{r=0}^{\infty} a_r(x)\, \omega^{\mu-r} \quad \text{as } r \to \infty. \tag{2}$$