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Lagrangian manifolds and the Maslov operator, by A. S. Mishchenko, V. E. Shatalov, and B. Yu. Sternin. (Translated from the Russian by Dana Mackenzie.) Springer-Verlag, Berlin, Heidelberg, New York, 1990, 395 pp., $69.50. ISBN 0-540-13613-4

From the foreword: "This book presents Maslov's canonical operator method for finding asymptotic solutions of pseudodifferential equations." Nothing in the foreword about Hörmander's theory of Fourier integral operators, not even Hörmander's name is mentioned in what the authors call "a more or less full list of works in which Maslov's theory is being developed or used." On the other hand, the book contains an appendix on Fourier-Maslov Integral Operators, in the introduction of which it is stated: "Moreover, it was later clarified that Maslov's canonical operator method, when applied in the situation of Fourier integral operators, precisely coincided with the latter."

If the subjects are the same, then it seems rather pointless to publish a translation of the Russian book written in 1978, in the presence of the very complete and clear exposition of the theory of Fourier integral operators in Volume 4 of Hörmander [H14]. If the subjects are not identical, then it is worthwhile to also have a presentation in English of Maslov's canonical operator method. In concordance with the authors of the book, I think that both assumptions hold.

In order to explain this, I would like to begin with a short description of the objects of study. Having these available, it will be better possible to compare the various theories and to describe the contents of the book in more detail.

High frequency waves are functions of the form

$$u(x, \omega) = e^{i\omega\phi(x)}\, a(x, \omega). \tag{1}$$

Here ω is a positive real number representing the *frequency*, and one is interested in asymptotic formulas as $\omega \to \infty$. The *phase function* $\phi(x)$ is a smooth real-valued function of the position variables $x = (x_1, \dots, x_n)$. The *amplitude* $a(x, \omega)$ is a complex-valued smooth function of x that has an asymptotic expansion in decreasing powers of ω of the form

$$a(x, \omega) \sim \sum_{r=0}^{\infty} a_r(x)\, \omega^{\mu - r} \quad \text{as } r \to \infty. \tag{2}$$

Such functions $u(x, \omega)$ occur as asymptotic solutions of equations involving linear partial differential operators $P = P(x, \partial/\partial x, \omega)$, with powers of ω in the coefficients. Here "asymptotic solution" means that it is only required that Pu is of order ω^{-N} as $\omega \to \infty$, for a suitably large N, instead of satisfying the exact equation $Pu = 0$. Such asymptotic solutions can be constructed locally for very general problems and are good approximations of exact solutions.

The asymptotic equation $Pu \sim 0$ is equivalent to a nonlinear first-order partial differential equation

$$p(x, d\phi(x)) = 0 \tag{3}$$

for the phase function, coupled with a successive system of linear first-order partial differential equations

$$\sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j}(x, d\phi(x)) \frac{\partial a_r}{\partial x_j} + c(x)\, a_r = F_r(a_0, \dots, a_{r-1})(x) \tag{4}$$

for the terms $a_r(x)$ in the expansion of the amplitude. Here $p = p(x, \xi)$ is a smooth function on the cotangent bundle T^*X of the x-space X; note that $\xi = d\phi(x)$ is a linear form on the tangent space at the point x. The function $p(x, \xi)$ is a polynomial in ξ of order m, determined by the operator P, and is called the *principal symbol* of P. We shall assume for the moment that p is real-valued. Furthermore, $F_0 = 0$ and F_r is a linear partial differential operator of order at most $\max(m, r+1)$ in $a_0, \dots, a_{r-1}$, defined by P and ϕ. Also, $c(x)$ is a smooth function only depending on P and ϕ. The strategy is to solve first ϕ from (3) and then successively solve a_0, a_1, etc., from the so-called *transport equations* (4).

The graph

$$\Lambda_\phi := \{(x, d\phi(x)) \mid x \in X\} \tag{5}$$

of the total derivative $d\phi$ of ϕ is a smooth n-dimensional submanifold of T^*X, with the special property that the canonical two-form

$$\sigma := \sum_{j=1}^{n} d\xi_j \wedge dx_j \tag{6}$$

of the cotangent bundle vanishes on Λ_ϕ. In general an n-dimensional submanifold Λ of T^*X on which $\sigma = 0$ is called a *Lagrange manifold.* The equation (3) means that $\Lambda = \Lambda_\phi$ is contained in the zeroset of p. For a Lagrange manifold this in turn implies that Λ is invariant under the flow of the *Hamilton system*

$$\frac{dx_j}{dt} = \frac{\partial p}{\partial \xi_j}(x, \xi), \qquad \frac{d\xi_j}{dt} = -\frac{\partial p}{\partial x_j}(x, \xi) \tag{7}$$

in T^*X, defined by the function p.

In general, if one prolongs Λ_ϕ along the solution curves of (7), then it will cease to be the graph of a function (ξ as a function of x). Usually at such points also the solution a_0 of the transport equation will blow up; these points correspond to the *caustics* of geometrical optics. In a neighborhood of such points, one still can find a prolongation of the asymptotic solution u, of the equation $Pu = 0$, by replacing the simple progressing wave (1) by an *oscillatory integral*

$$u(x, \omega) = \omega^{k/2} \int e^{i\omega\phi(x, \theta)} a(x, \theta, \omega)\, d\theta. \tag{8}$$

That is, a continuous superposition of waves over auxiliary variables $\theta = (\theta_1, \dots, \theta_k) \in \mathbf{R}^k$. The *principle of stationary phase* tells that the only contributions to the integral that are not rapidly decreasing as $\omega \to \infty$, come from the set

$$S_\phi := \left\{(x, \theta) \mid \frac{\partial \phi}{\partial \theta}(x, \theta) = 0\right\}, \tag{9}$$

the set of points where the phase function ϕ is stationary with respect to the integration variables θ. This time one takes

$$\Lambda_\phi := \left\{\left(x, \frac{\partial \phi}{\partial x}(x, \theta)\right) \mid (x, \theta) \in S_\phi\right\}. \tag{10}$$

The phase function ϕ is called *nondegenerate* if $d_{(x, \theta)} d_\theta \phi$ has maximal rank equal to k. In this case Λ_ϕ is a Lagrange submanifold of T^*X. If one arranges that it is equal to the prolongation of the previously defined one along the solution curves of (7), then a suitable choice of the amplitude a will lead to a prolongation of the asymptotic solution of $Pu = 0$.

The main results of the theory now are as follows. Every Lagrange submanifold Λ of T^*X is locally of the form $\Lambda = \Lambda_\phi$ for a suitable nondegenerate phase function ϕ. There is an invariantly defined smooth line bundle L over Λ and, for each nondegenerate phase function ϕ, an isomorphism i_ϕ from the space of smooth functions on S_ϕ to the space of smooth sections of L, with the following property. Suppose that ϕ and ψ are nondegenerate phase functions such that locally $\Lambda_\phi = \Lambda_\psi = \Lambda$. Let u be defined by (8) and v by a similar expression, with ϕ and a replaced by ψ and b, respectively. Then $u - v$ is equal to such an integral with an amplitude of one degree lower in ω, if and only if

$$i_\phi(a_0|_{S_\phi}) = i_\psi(b_0|_{S_\psi}). \tag{11}$$

This can be rephrased as follows. Let $I^\mu(\Lambda)$ denote the space of oscillatory functions $u(x, \omega)$, which locally can be written as (8), with an amplitude a of order μ, and phase function ϕ such that, locally, $\Lambda_\phi = \Lambda$. Then the local sections (11) of L piece together to a global section of L, called the *symbol* $s_\mu(u)$ *of* u *of order* μ. The symbol map $u \mapsto s_\mu(u)$ induces an isomorphism

$$s_\mu \colon I^\mu(\Lambda)/I^{\mu-1}(\Lambda) \xrightarrow{\sim} \Gamma(\Lambda, L). \tag{12}$$

Now suppose that the Lagrange manifold Λ is contained in the zeroset of p; recall that this implies Λ is invariant under the flow of (7). The operator P induces a connection in L, defined locally by the function $c(x)$ in (4). If $u \in I^\mu(\Lambda)$, then $Pu \in I^{\mu+m-1}$ and the symbol of Pu of order $\mu + m - 1$ is equal to $1/i$ times the covariant derivative of $s_\mu(u)$ with respect to the Hamilton vector field H_p of p on Λ. So if we start with a symbol that is covariantly constant along the H_p-curves, and then succesively add correction terms $u_j \in I^{\mu-j}$ for which the covariant derivative is chosen appropriately, then we arrive at $u \in I^{\mu}(\Lambda)$ such that $Pu \in I^{\mu-N}(\Lambda)$ for arbitrarily large N. In this way the construction of asymptotic solutions (1) can be globalized in a way that includes caustic points.

If one also integrates over the frequency variable, then the integral converges in distribution sense. In this case the phase function is homogeneous of degree 1 in the integration variables, and as a consequence, the Lagrange manifold Λ becomes conic with respect to the ξ-variables. Its projection to the x-space is a hypersurface S, possibly with singularities and the distribution $u(x)$, called a *Lagrange distribution*, has singularities along S, of a very specific nature.

Another variation of the theory occurs if one allows the phase function to be *complex valued*, but with nonegative imaginary part, in order to exclude exponential growth. This generalization has applications in many problems, both in the case of oscillatory functions and in the case of Lagrange distributions. This concludes the review of the theory.

The introduction of oscillatory functions defined globally by means of Lagrange manifolds is due to Maslov, in his 1965 book *Theory of perturbations and asymptotic methods.* The oscillatory integrals are obtained by performing a Fourier transform, in a part of the variables, to a simple progressing wave. In terms of phase functions, this means that only phase functions of a restricted type are used, e.g., linear in the Fourier transform variables.

Maslov's applications and terminology are oriented towards quantum mechanics: the frequency variable is denoted by $1/h$, with $h =$ Planck's constant, the simple progressing waves are called WKB approximations, and the phase functions are called actions. The latter reminds of the Feynman path integral, which is an oscillatory integral over an infinite-dimensional space of curves γ, with phase function at γ equal to the action integral over γ. If I understand it correctly, Maslov's *canonical operator* is the assignment of the oscillatory function modulo lower order ones, to a section of the line bundle L over Λ. That is, it is equal to the inverse of the symbol map s_{μ} in (12). The adjective "canonical" seems to refer to the standard form in which the differential equation can be brought at certain caustics, like with the Airy function, and not to the canonical symplectic structure of the cotangent bundle.

Maslov also gave applications to difference schemes and, in the distributional version, to the propagation of singularities of solutions of hyperbolic equations. His extension to complex phase functions was announced at the International Congress in Nice in 1970.

In 1971 Hörmander published his calculus of *Fourier integral operators,* which are defined as integral operators, for which the distribution kernels are equal to Lagrange distributions. The corresponding conic Lagrange manifold Λ in the cotangent bundle of the product space can then be viewed as a homogeneous canonical transformation Φ between the respective cotangent bundles. This has the advantage that the composition of two Fourier integral operators with canonical transformations Φ_1 and Φ_2, respectively, is a Fourier integral operator with canonical transformation equal to the composition of Φ_1 and Φ_2. Hörmander's theory was more restricted than Maslov's in that it did not treat explicitly the asymptotic solutions defined by oscillatory functions; but on the other hand, it clarified and completed the calculus at many points.

From that time on, Fourier integral operators were widely recognized as an extremely flexible tool in the analysis of linear partial differential operators.

A prime example of this is Egorov's observation in 1969, using Maslov's theory, that many partial differential operators can be brought into an extremely simple standard form, using a conjugation by a suitable Fourier integral operator. The theory of Fourier integral operators with complex phase functions, in Hörmander's style, was developed by Melin and Sjöstrand in 1974–76.

The book of Mishchenko e.a. is a textbook on Maslov's theory. It starts with an introduction in which applications to partial differential equations are the point of departure. Because here the theory is not yet available, there are quite sudden jumps from very elementary considerations to conclusions that can at best be understood in a very intuitive manner. It is followed by Part I, consisting of three chapters. This is a systematic exposition of the differential geometric theory of Lagrange manifolds, including the case that is connected with complex phase functions. Part II starts with three chapters in which Maslov's canonical operator is presented, in the asymptotic, oscillatory function version. Part II is concluded with a chapter on applications. It is my impression that Parts I and II are clearly written and form a useful textbook on the graduate student level.

The aforementioned appendix treats Fourier integral operators in terms of Maslov's canonical operator. Although the authors consider this "much more transparent, geometric, and allowing one to obtain the answer in a more finished form" than Hörmander's Fourier integral operators, I nevertheless would like to advise a student to study also Hörmander's exposition in Chapter 25 of [H14], if only because this is the standard framework in much of the research literature in linear partial differential equations.

In general the English text has a pleasant style. Because I do not know Russian, I cannot compare the English translation with the Russian original, so at the few places where I wondered what really was meant, I do not know whether this was caused by the translation or by the original text. For instance, in the foreword, it is stated that "the fundamental results pertaining to the decomposition of a rapidly oscillating function by the stationary phase method *belong* to M. V. Fedoriuk [1]–[3]"; an exaggerating possessive form. Another example is the statement on p. 28 in the introduction, that "*any* asymptotic solution of a linear partial differential equation is given by a Lagrange manifold." The Lagrange ones actually form a very special, nice subclass of asymptotic solutions, which at best are the building blocks for the general solution. But despite my critical remarks, the overall impression is that this is a useful, clear, and pleasantly written textbook.

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