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Connections, definite forms and four-manifolds, by T. Petrie and J. Randall. Clarendon Press, Oxford, 1990, 130 pp. , $45.00. ISBN 0-19-853599-6

One of the most important classes of objects in topology is the class of manifolds. These objects appear naturally in fields ranging from geometry and analysis to physics and mathematical economics. The ubiquity of manifolds suggests the importance of answering the most basic questions about them, namely, existence and uniqueness: For a given set of algebraic topological data, is there a manifold realizing this data? If so, how many are there, and how can they be classified? Much progress has been made on these questions. In dimensions two and below, a complete classification of manifolds has been known for many years. In dimensions three and higher, the situation is much more complicated. One might expect manifold theory to become increasingly harder with increasing dimension, but in fact, the history of manifold topology suggests otherwise. In dimensions five and higher, there is enough room to prove several powerful theorems (in particular, Smale's h-Cobordism Theorem [M] and the Surgery Theorem [W]). As a result, the most basic questions about manifolds in dimensions five and higher were largely reduced to (hard) algebraic topology in the 1960s. Consequently, dimensions three and four have been the most enigmatic. Although basic questions remain unanswered in dimension three, much progress has been made, partly through the close connection with dimension two, where a simple classification exists. Dimension four, however, was virtually impenetrable until the last decade, and even now it remains largely mysterious.

To understand the state of research in dimension four, we must be aware of a distinction that appears in high-dimensional theory. There are actually several different settings for studying manifolds. We can consider our manifolds to be purely *topological*, that is, certain topological spaces to be classified up to homeomorphism. Alternatively, we may consider *smooth* manifolds, which have been endowed with extra structure allowing us to do calculus. While topological manifolds may seem more fundamental to topologists, it is the smooth manifolds that tend to appear in applications. The distinction between smooth and topological manifolds turns out to be insignificant in dimensions three and below. In higher dimensions, however, there are topological manifolds that are unsmoothable and others that admit several nondiffeomorphic smooth structures. The study of the interplay between these two categories, or *smoothing theory*, was developed for dimensions five and up in the late 1960s [KS]. The final result is that in high dimensions the smooth and topological theories are quite similar, and the relation between the two is determined by algebraic topology.

The theory of 4-manifolds essentially began with two major breakthroughs in the early 1980s. The first breakthrough was Freedman's work [F, FQ] in the topological category. Freedman showed that topological 4-manifolds behave like their higher dimensional counterparts (at least, when the fundamental group is not too large). In particular, for closed (i.e., compact and without boundary), simply connected topological 4-manifolds, Freedman achieved a complete classification in terms of the intersection form (or equivalently, cup product pairing) in the middle dimension, and one other $\mathbb{Z}_2$ invariant. (This is essentially the only algebraic data that a closed, simply connected 4-manifold carries.) For all of its power and beauty, however, Freedman's work said nothing about the smooth category, the world of geometry, analysis, and physics. The breakthrough for smooth 4-manifolds came from Donaldson's revolutionary ideas from gauge theory. Donaldson's work reveals that smooth 4-manifolds are much *different* from their high-dimensional counterparts. The main theorems from high dimensions fail catastrophically in this setting. In particular, when we compare with Freedman's work we find that smoothing theory in dimension four is radically different from the relatively manageable, algebraic theory from high dimensions—indeed, it is much more complex. For example, dimension four is the only dimension in which $\mathbb{R}^n$ admits exotic smooth structures (i.e., manifolds homeomorphic to $\mathbb{R}^n$ but not diffeomorphic to it). In fact, more recent work of Taubes [T2] shows that there are uncountably many diffeomorphism types, and these can occur in continuous families—in stark contrast to the discrete nature of high-dimensional smoothing theory (see also [G]).

Donaldson's ideas have led to a deluge of new theorems about smooth 4-manifolds. His first theorem [D1] asserted that if a smooth, closed, simply connected 4-manifold has a definite intersection form, then the form must be diagonalizable. Since there are many nondiagonalizable, definite forms over the integers, and Freedman asserts that these are realized topologically, we obtain many unsmoothable 4-manifolds, some of which "should" be smoothable by the predictions of higher dimensional smoothing theory. The failure of the smooth surgery theorem and the existence of exotic $\mathbb{R}^4$'s are corollaries. Subsequently, Fintushel and Stern greatly simplified Donaldson's proof [FS1] and applied his methods to obtain information about the homology cobordism group of homology 3-spheres [FS2]. (More recently, Furuta [Fu] has expanded this work and showed that the group is not finitely generated.) Donaldson has also constructed powerful new invariants for smooth 4-manifolds. Initially [D2], he used these to distinguish two simply connected algebraic surfaces (smooth 4-manifolds) that were homeomorphic but not diffeomorphic. These "should" have been diffeomorphic, by high-dimensional smoothing theory, and they provided a counterexample to the smooth h-cobordism theorem. Subsequently, Friedman and Morgan [FM1, FM2] used Donaldson's invariants to distinguish infinite families of such homeomorphic (and h-cobordant) examples. This contrasts with high-dimensional smoothing theory, in which a compact manifold admits only finitely many smooth structures. Various other questions about simply connected algebraic surfaces have also fallen: Donaldson [D3] showed that they cannot split as connected sums of manifolds with indefinite forms, and simply connected 4-manifolds have been constructed that are irreducible under connected sum but not diffeomorphic to algebraic surfaces [GM]. The underlying theme of all of these results is that the theory of smooth 4-manifolds (even in the simply connected case) is much more complex than anyone had realized, and this bountiful structure is likely to provide for active research for years to come.

Donaldson's work centers on the *moduli space* $\mathscr{M}$ of *self-dual connections* on a bundle over a 4-manifold M. These are solutions to a certain nonlinear partial differential equation, which we briefly describe, although at the present level of discourse the details are hardly necessary. We begin with a G-vector bundle E over a Riemannian manifold M. In principle, G can be any compact Lie group, but in practice it is usually $SU(2)$ or $SO(3)$. The set of G-connections on E forms an affine space $\mathscr{C}$. Each connection in $\mathscr{C}$ has a curvature form that is a 2-form on M with coefficients in the bundle $\mathfrak{g}^E$ of Lie algebras associated to E by the adjoint action. The Hodge star operator provides an involution on the space of 2-forms, and hence, on 2-forms with coefficients in $\mathfrak{g}^E$. A connection is called *self-dual* if its curvature lies in the fixed set of the star operator. Self-dual connections are described by a first-order nonlinear equation. The set of all self-dual connections in $\mathscr{C}$ is infinite dimensional because E admits a large symmetry group that preserves self-duality. Specifically, if $\mathscr{G}$ denotes the group of all G-bundle automorphisms (or *gauge transformations*) of E, then $\mathscr{G}$ acts on $\mathscr{C}$ and preserves self-duality. The quotient space $\mathscr{C}/\mathscr{G}$ is (after a suitable completion) a singular Hilbert manifold with well-understood algebraic topology. The *moduli space* $\mathscr{M} \subset \mathscr{C}/\mathscr{G}$ is the quotient by $\mathscr{G}$ of the set of self-dual connections. In this setting, the self-duality equation is essentially elliptic, so after a suitable perturbation of the equation (or the metric on M [FU]) $\mathscr{M}$ becomes a finite-dimensional singular manifold. The singularities are well understood. They come from singularities of $\mathscr{C}/\mathscr{G}$, which correspond to *reducible* connections. In the situations we are considering, $G = SU(2)$ or $SO(3)$, the singularities correspond to $U(1)$ or $SO(2)$ reductions, and each will have a neighborhood in $\mathscr{M}$ that is a cone on a complex projective space.

We can now see how Donaldson's machinery works. To prove his first theorem, Donaldson began with a hypothetical 4-manifold with a nondiagonalizable intersection form. He studied the moduli space $\mathscr{M}$ associated to the simplest nontrivial $SU(2)$ bundle over M. This turned out to be a singular 5-manifold. $\mathscr{M}$ was not compact, but by using some sophisticated analysis of Taubes [T1], Donaldson showed the end of $\mathscr{M}$ was collared by $M \times \mathbb{R}$, so $\mathscr{M}$ could be compactified by adding a copy of the original manifold M. He then deleted the singularities to obtain a compact, oriented 5-manifold $\widehat{\mathscr{M}}$ bounded by M and some number of copies of $\mathbb{C}P^2$. The number was determined by the intersection form of M, and simple arithmetic showed that the boundary of $\widehat{\mathscr{M}}$ had nonzero signature. Since any 4-manifold that bounds an oriented 5-manifold must have signature zero, this gave the required contradiction.

The method of Fintushel and Stern also involved cobordism, but the details were much simpler. In place of an $SU(2)$ bundle, Fintushel and Stern used an $SO(3)$ bundle. They chose their bundle so that $\mathscr{M}$ was compact. This avoided the hard analysis of the Collar Theorem. After deleting the reducible connections, they obtained a compact manifold $\widehat{\mathscr{M}}$, whose boundary was an odd number of copies of complex projective space. This led to a contradiction without any discussion of orientability of $\widehat{\mathscr{M}}$. (In the simplest case, $\widehat{\mathscr{M}}$ was a compact one-manifold bounded by an odd number of points—clearly an impossible object.)

Donaldson's invariants can also be seen in our description of $\mathscr{M} \subset \mathscr{C}/\mathscr{G}$. Ignoring the myriad technical details (such as the fact that $\mathscr{M}$ is frequently noncompact), we may think of $\mathscr{M}$ as representing a homology class in $\mathscr{C}/\mathscr{G}$.

(More precisely, if the intersection form of M is indefinite, we obtain a class in $\mathscr{C}^*/\mathscr{G}$, the nonsingular Hilbert manifold consisting of the complement of the reducible connections in $\mathscr{C}/\mathscr{G}$.) Under suitable algebraic hypotheses this idea can be made precise, and the homology class can be shown to be independent of the metric on M. Since the ambient space has large (and well-understood) homology, we obtain a large family of diffeomorphism invariants (depending on our choices of G and E). Various methods have been developed for computing these invariants (e.g., [D2, D3, FM1, FM2, GM]), resulting in a plethora of spectacular theorems about smooth 4-manifolds.

Connections, definite forms and four-manifolds, by Petrie and Randall, provides a (relatively) elementary introduction to Donaldson theory. The book centers on gauge theory with $SO(3)$-connections and provides a good exposition of Donaldson's first theorem from the Fintushel-Stern viewpoint. This is followed by a brief exposition of $\mathbb{Z}_n$-equivariant gauge theory and the Fintushel-Stern work on cobordism of homology 3-spheres. Since there are already references for Donaldson's first theorem from his original $SU(2)$ viewpoint [FU, L], the main advantage of this new book lies in its relative simplicity. By working with $SO(3)$, the authors have avoided dealing with the Collar Theorem, thereby eliminating much difficult analysis. The discussion of orientability has been bypassed, and even the delicate question of existence of self-dual connections has been finessed by the Fintushel-Stern approach of building the bundle E so that it explicitly admits a self-dual $SO(2)$-connection. The book also includes much more elementary background material than previous references, which is an important advantage in a field such as this that depends on a wide range of mathematics. Altogether, this is perhaps the best textbook currently available for a one-semester graduate course introducing Donaldson theory. The principal drawback of the book is that it makes virtually no mention of recent developments such as Donaldson's invariants and their applications. (This seems to be the inevitable consequence of dealing with an area that continues to grow explosively). On the other hand, any reasonable treatment of Donaldson's invariants would require the introduction of much additional machinery, so perhaps this is best left to more advanced books. In any case, more comprehensive references exist, notably [DK].

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ROBERT E. GOMPF
UNIVERSITY OF TEXAS, AUSTIN

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Degeneration of Abelian Varieties, by Gerd Faltings and Ching-Li Chai. Ergeb. Math. Grenzgeb. (3), vol. 22, Springer-Verlag, New York, 1990, 316 pp., $39.80. ISBN 3-540-52015-5

The publication of this book was an event long awaited by specialists. The algebraic compactification of the moduli of principally polarized abelian varieties is the culmination of a long and fruitful line of research in algebraic geometry. Before turning to the contribution of the book proper, we will summarize some of the principal events in the history of the subject.

Abelian varieties begin with elliptic curves. An elliptic curve over an algebraically closed field k is determined, up to nonunique isomorphism, by its j-invariant, which can be an arbitrary element of k. One says that the affine line over $\mathbf{Z}$, with coordinate j, is a *moduli variety* for elliptic curves: the word "moduli" here means that j is a parameter. This moduli variety has a natural compactification: the projective line over $\mathbf{Z}$. As j tends to infinity, the corresponding elliptic curve degenerates to a singular cubic. Among singular cubics, the least degenerate are those with an ordinary double point. It is therefore natural to complete the modular picture by making the singular cubics correspond to the exceptional value $j = \infty$. We will come back to this problem further on. For the moment, let us say that the contribution of Chai and Faltings is to extend this type of construction to higher dimensional abelian varieties.