BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, July 1992

A CLASS OF NONSYMMETRIC HARMONIC RIEMANNIAN SPACES

EWA DAMEK AND FULVIO RICCI

ABSTRACT. Certain solvable extensions of H-type groups provide noncompact counterexamples to a conjecture of Lichnerowicz, which asserted that "harmonic" Riemannian spaces must be rank 1 symmetric spaces.

A Riemannian space M with Laplace-Beltrami operator Δ is called *harmonic* if, given any function $f(x)$ on M depending only on the distance $d(x, x_0)$ from a given point x_0, then also $\Delta f(x)$ depends only on $d(x, x_0)$.

Equivalently, M is harmonic if for every $p \in M$ the density function $\omega_{x_0}(x)$ expressed in terms of the normal coordinates around the point x_0 is a function of $d(x, x_0)$ (see [1, 11]).

In 1944 Lichnerowicz [10] proved that in dimensions not greater than 4 the harmonic spaces coincide with the rank-one symmetric spaces. He also raised the question of determining whether the same is true in higher dimensions.

Among the most recent progress made on the so-called Lichnerowicz conjecture, Szabó [11] proved it to hold true in arbitrary dimension for compact manifolds with finite fundamental group.

In this announcement we present a counterexample that arises in the noncompact case. It proves the Lichnerowicz conjecture not to be true in general for infinitely many dimensions, the smallest of them being 7.

This example is based on the notion of H-type group due to Kaplan [8], and on the geometric properties of their one-dimensional extensions S introduced by Damek [5] and studied also in [2, 3, 4, 6].

The rank-one symmetric spaces of the noncompact type different from the hyperbolic spaces are special examples of such groups S. Even though symmetry is a main geometric difference that distinguishes some "good" S from other "bad" S, it turns out that a large part of the harmonic analysis on these groups can be worked out regardless of this distinction.

A detailed account of this is given in our forthcoming paper [7].

We thank J. Faraut and A. Korányi for calling our attention to the Lichnerowicz conjecture.

1. THE EXTENSION S OF AN H-TYPE GROUP

An H-type (or Heisenberg-type) algebra is a two-step nilpotent Lie algebra $\mathfrak{n}$ with an inner product $\langle \ , \ \rangle$ such that if $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v} = \mathfrak{z}^\perp$, then the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ given by

$$\langle J_Z X, Y\rangle = \langle [X, Y].Z\rangle$$

for $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, satisfies the identity $J_Z^2 = -|Z|^2 I$ for every $Z \in \mathfrak{z}$.

Received by the editors July 11, 1991.
1991 *Mathematics Subject Classification*. Primary 53C25, 53C30, 43A85, 22E25, 22E30.

An H-type group is a connected and simply connected Lie group N whose Lie algebra is H-type.

Let $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}T$ be the extension of $\mathfrak{n}$ obtained by adding the rule

$$[T, X+Z] = \tfrac{1}{2}X + Z$$

for $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. Let $S = NA$ be the corresponding connected and simply connected group extension of N, where $A = \exp_S(\mathbb{R}T)$. We parametrize the elements of S in terms of triples $(X, Z, a) \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}^+$ by setting

$$(X, Z, a) = \exp_S(X+Z)\exp_S(\log a\, T).$$

The product on S is then

$$(X, Z, a)(X', Z', a') = (X + a^{1/2}X', Z + aZ' + \tfrac{1}{2}a^{1/2}[X, X'], aa').$$

We introduce the inner product

$$\langle X + Z + tT | X' + Z' + t'T\rangle = \langle X|X'\rangle + \langle Z|Z'\rangle + tt'$$

on $\mathfrak{s}$ and endow S with the induced left-invariant Riemannian metric.

Proposition 1 [4, 5, 9]. *Let $M = G/K$ be a rank-one symmetric space of the noncompact type different from $SO_e(n, 1)/SO(n)$, and let $G = NAK$ be the Iwasawa decomposition of G. Then N is an H-type group and the map $s \mapsto sK$ is an isometry from $S = NA$ to G/K.*

From the classification of symmetric spaces one knows that if the H-type group N appears in the Iwasawa decomposition of a rank-one symmetric space, then the dimension of its center equals 1, 3, or 7. On the other hand, there exist H-type groups with centers of arbitrary dimensions [8]. Direct proofs can be given [4, 5] to show that the space S is symmetric if and only if N is an "Iwasawa group."

We can then conclude that *there are infinitely many S that are not symmetric.*

2. The volume element in radial normal coordinates

Given unit elements $X_0 \in \mathfrak{v}$ and $Z_0 \in \mathfrak{z}$, we denote by S_{X_0, Z_0} the 4-dimensional subgroup of S generated by X_0, Z_0, $J_{Z_0}X_0$, and T. Clearly, S_{X_0, Z_0} is also the extension of an H-type group, precisely of the 3-dimensional Heisenberg group.

Proposition 2 [3]. *S_{X_0, Z_0} is a totally geodesic submanifold of S, isometric to the Hermitian symmetric space $SU(2, 1)/S(U(2) \times U(1))$.*

This observation suggests two alternative realizations of S [4, 7]:

(1) the "Siegel domain" model:

$$D = \{(X, Z, t) \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R} : t > \tfrac{1}{4}|X|^2\}$$

with the metric transported from S via the map $h(X, Z, a) = (X, Z, a + \frac{1}{4}|X|^2)$;

(2) the "ball" model:

$$B = \{(X, Z, t) \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R} : |X|^2 + |Z|^2 + t^2 < 1\}$$

with the metric transported from D via the inverse of the "Cayley transform" $C : B \to D$ given by

$$C(X, Z, t) = \frac{1}{(1-t)^2 + |Z|^2}\left(2(1 - t + J_Z)X\ ,\ 2Z\ ,\ 1 - t^2 - |Z|^2\right).$$

The map $C^{-1} \circ h : S \to B$ maps the identity element $e = (0, 0, 1)$ to the origin and its differential at e equals $\frac{1}{2}I$.

Let $m = \dim \mathfrak{v}$, $k = \dim \mathfrak{z}$. The number $Q = \frac{m}{2} + k$ is called the homogeneous dimension of N. It is easily checked that

$$dm_L = a^{-Q-1}\, dX\, dZ\, da$$

is a left Haar measure on S and is also the Riemannian volume element.

Applying Proposition 2, using the properties of the Cayley transform in the symmetric Hermitian case, and explicitely computing the Jacobians of h and C, we obtain the following result.

Theorem 1 [7]. (1) *In the metric on B transported from S via $C^{-1} \circ h$, the geodesics through the origin are the diameters and the distance ρ from a point $(X, Z, t) \in B$ to the origin is a function of $r = (|X|^2 + |Z|^2 + t^2)^{1/2}$, namely,*

$$\rho = \log \frac{1+r}{1-r}\,.$$

(2) *Introducing polar coordinates (r, ω) on B and denoting by $d\sigma(\omega)$ the surface measure on S^{m+k}, the volume element on B is given by*

$$2^{m+k+1}(1-r^2)^{-Q-1} r^{m+k}\, dr\, d\sigma(\omega) = 2^{m+k}\left(\cosh\frac{\rho}{2}\right)^k \left(\sinh\frac{\rho}{2}\right)^{m+k} d\rho\, d\sigma(\omega)\,.$$

The coordinates (ρ, ω) are the radial normal coordinates on B around the origin. Then (2) proves that the volume density depends only on ρ. The same is true for normal coordinates around any other point, since S and B are isometric and S is obviously homogeneous. We can then conclude that

Corollary. *For every H-type group N, $S = NA$ is a harmonic space.*

Consider now the Laplace-Beltrami operator Δ on S. Its extension to the domain

$$\mathscr{D}(\Delta) = \{f \in L^2(S, dm_L) : \Delta f \in L^2(S, dm_L)\}$$

is selfadjoint and positive. From the spectral resolution

$$\Delta = \int_0^{+\infty} \lambda\, dE_\lambda$$

one constructs the heat semigroup

$$T_t = e^{-t\Delta} = \int_0^{+\infty} e^{-t\lambda}\, dE_\lambda\,.$$

Since Δ is left-invariant,

$$(T_t f)(x) = (f * p_t)(x) = \int_S f(y) p_t(y^{-1}x)\, dm_L(y)$$

and $p(t, x) = p_t(x)$ is a smooth function on $\mathbb{R}^+ \times S$. In [7] we also prove the following strong-harmonicity property of S.

Theorem 2. *The functions $p_t(x)$ depend only on the geodesic distance from x to e.*

3. Dimensions of the nonsymmetric S

A complete list of H-type groups can be obtained from the classification of Clifford modules [8]. For small values of $k = \dim \mathfrak{z}$ we write the dimensions of the corresponding nonsymmetric S:

k	$\dim S$
1	—
2	$7 + 4n$
3	$12 + 4n$
4	$13 + 8n$
5	$14 + 8n$
6	$15 + 8n$
7	$24 + 8n$
8	$25 + 16n$

where $n = 0, 1, \dots$.

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Instytut Matematyczny Uniwersytetu Wrocławskiego, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

E-mail address: FRICCI@POLITO.IT