

GENERALIZING THE HYPERBOLIC COLLAR LEMMA

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ABSTRACT. We discuss two generalizations of the collar lemma. The first is the stable neighborhood theorem which says that a (not necessarily simple) closed geodesic in a hyperbolic surface has a “stable neighborhood” whose width only depends on the length of the geodesic. As an application, we show that there is a lower bound for the length of a closed geodesic having crossing number k on a hyperbolic surface. This lower bound tends to infinity with k .

Our second generalization is to totally geodesic hypersurfaces of hyperbolic manifolds. Namely, we construct a tubular neighborhood function and show that an embedded closed totally geodesic hypersurface in a hyperbolic manifold has a tubular neighborhood whose width only depends on the area of the hypersurface (and hence not on the geometry of the ambient manifold). The implications of this result for volumes of hyperbolic manifolds is discussed. We also derive a (hyperbolic) quantitative version of the Klein-Maskit combination theorem (in all dimensions) for free products of fuchsian groups. Using this last theorem, we construct examples to illustrate the qualitative sharpness of the tubular neighborhood function.

1. INTRODUCTION

It is natural to look for universal properties of discrete subgroups of a fixed lie group. The lie group we are concerned with is the orientation preserving isometries of real hyperbolic space $\text{Isom}^+(\mathbb{H}^n)$. A fundamental universal property in dimension three is the Jorgensen inequality [Jo]. Another celebrated property is the Margulis lemma [Th] (actually, this lemma holds for a bigger class of lie groups). We would like to focus on the well-known collar lemma due to Keen [Ke]; the sharp form is due to Buser [Bu]. There are many different versions of this lemma in the literature (cf. [Ba1, Be, Ha, M1, Ma, R]).

Let $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function $r(x) = \log \coth(x/2)$. Observe that this function monotonically decreases to zero and satisfies $r^2 = 1$. The collar lemma says that any simple closed geodesic of length ℓ on a hyperbolic surface (a complete surface of constant curvature negative one) has a collar neighborhood of width $r(\frac{\ell}{2})$. Furthermore, a collection of disjoint simple closed geodesics have disjoint collars. This result is universal in the sense that the collar width does not depend on the underlying hyperbolic structure of the surface, only on the length of the geodesic.

We would like to discuss two generalizations of the collar lemma. In the first generalization (the stable neighborhood theorem), we remove the hypothesis in the collar lemma that the closed geodesic be simple. The concept of a collar is then replaced by the notion of a stable neighborhood. The second generaliza-

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tion of the collar lemma involves totally geodesic hypersurfaces in hyperbolic manifolds (complete riemannian manifolds of constant curvature negative one). Simple closed geodesics are replaced by totally geodesic embedded hypersurfaces and the notion of a collar is then replaced by a tubular neighborhood.

2. THE STABLE NEIGHBORHOOD THEOREM

The *neighborhood of width d* , $U_d(\omega)$, about a geodesic ω in a hyperbolic surface is the set of all points within a distance d from the geodesic. Suppose ω is a closed geodesic. Then the neighborhood $U_d(\omega)$ is said to be *stable* if for any two connected (smooth) lifts of ω , say ω_1 and ω_2 , we have

$$\omega_1 \cap \omega_2 \neq \emptyset \text{ if and only if } U_d(\omega_1) \cap U_d(\omega_2) \neq \emptyset.$$

Theorem 1 (The Stable Neighborhood Theorem). *A closed geodesic of length ℓ on a hyperbolic surface has a stable neighborhood of width $r(\frac{\ell}{2})$. Furthermore, two disjoint closed geodesics ω_1 and ω_2 on the surface having lengths ℓ_1 and ℓ_2 have disjoint stable neighborhoods of widths $r(\frac{\ell_1}{2})$ and $r(\frac{\ell_2}{2})$ respectively, if ω_1 and ω_2 are separated by a disjoint union of simple closed geodesics.*

The above separation condition is necessary, since there are examples of closed geodesics that get arbitrarily close to any fixed boundary geodesic on a pair of pants.

Hempel, Nakanishi, and Yamada [He, N, Y1, Y2]) have independently shown that there is a universal lower bound for the length of a nonsimple closed geodesic on a hyperbolic surface. As a consequence of the stable neighborhood theorem we have

Corollary 2. *There exists an increasing sequence M_k (for $k = 1, 2, 3, \dots$), tending to infinity so that if ω is a closed geodesic with self-intersection number k , then $\ell(\omega) > M_k$. Thus the length of a closed geodesic gets arbitrarily large as its self-intersection gets large.*

3. THE TUBULAR NEIGHBORHOOD THEOREM

In [Ba2], it was shown that associated to any totally geodesic hypersurface in a hyperbolic manifold there exists a spectrum of numbers called the orthogonal spectrum. This spectrum is essentially the lengths of orthogonals that start and end in the hypersurface. The following can be thought of as a geometric study of the first element in that spectrum.

Let $V_n(r)$ be the volume of the n -dimensional hyperbolic ball of radius r . The function

$$c_n(A) = \frac{1}{2}(V_{n-1} \circ r)^{-1}(A)$$

is called the *n -dimensional tubular neighborhood function*. Observe that this function is monotone decreasing and tends to zero as A goes to infinity. The *area* of a hypersurface is with respect to the induced metric from the ambient hyperbolic manifold.

The following theorem shows that a closed embedded totally geodesic hypersurface in a hyperbolic manifold has a tubular neighborhood whose width only depends on the area of the hypersurface.

Theorem 3 (The Tubular Neighborhood Theorem). *Suppose M^n is a hyperbolic manifold containing Σ , an embedded closed totally geodesic hypersurface of area A . Then Σ has a tubular neighborhood of width $c_n(A)$. That is, the set of points*

$$\{x \in M : d(x, \Sigma) < c_n(A)\}$$

is isometric to the product $\Sigma \times (-c_n(A), c_n(A))$.

Furthermore, any disjoint set of such hypersurfaces has disjoint tubular neighborhoods.

The main idea in the proof of the tubular neighborhood theorem is that the hypersurface in question must contain an embedded disc of radius $r(d)$, where d is the length of the shortest common orthogonal from Σ to itself.

It is well known that there exists a constant a , so that if M is a hyperbolic three manifold containing K rank two cusps then the volume of M is bigger than Ka (see [Th]). Analyzing volumes of tubular neighborhoods, we can prove a version of this result for totally geodesic hypersurfaces in hyperbolic manifolds of all dimensions. Specifically, we have

Theorem 4. *There exist positive constants a_n , for each $n \geq 3$, depending only on the dimension n , so that if M^n is a hyperbolic manifold containing K closed embedded disjoint totally geodesic hypersurfaces then,*

$$\text{Vol}(M) > Ka_n.$$

For hyperbolic three manifolds, a_3 can be taken to be $\pi \left(\log 2 + \frac{\sqrt{2}}{2} \right)$, which is approximately 4.4.

In dimensions $n \geq 4$, if M_i^n is a sequence of (not necessarily distinct) hyperbolic n -manifolds each containing an embedded totally geodesic closed hypersurface of area A_i , where $A_i \rightarrow \infty$, then the volumes of the tubular neighborhoods of width $c_n(A_i)$ tend to infinity. In particular, the volumes $\text{Vol}(M_i) \rightarrow \infty$.

In dimension three, there exist examples of totally geodesic surfaces in hyperbolic three manifolds whose areas get arbitrarily large, but whose best possible tubular neighborhoods have bounded volume.

The main lemma needed to prove the lower volume bound in Theorem 4 is the rate of growth lemma.

Lemma (Rate of growth lemma). *\mathcal{V}_n , the volume of a (one-sided) tubular neighborhood of width $c_n(x)$ about a hypersurface having area x , has the following behavior at infinity,*

$$\lim_{x \rightarrow \infty} \mathcal{V}_n(x) = \begin{cases} \infty & \text{for } n \geq 4, \\ \pi & \text{for } n = 3, \\ 0 & \text{for } n = 2. \end{cases}$$

\mathcal{V}_3 is monotone increasing and \mathcal{V}_2 is monotone decreasing.

We remark that Kojima and Miyamoto [KM] have found the smallest volume hyperbolic three manifolds with totally geodesic boundary. This volume is about 6.45.

In dimension three, a rank two cusp has a cusp neighborhood whose boundary is a flat torus having shortest closed geodesic of length one. Furthermore, if the hyperbolic three manifold has more than one cusp, these regions will be disjoint.

In fact, this is the basis of Meyerhoff's lower volume bound for a hyperbolic three manifold containing rank two cusps [Me].

The next corollary extends this to our setting:

Corollary 5. *The tubular neighborhood of width $c_3(A)$ about a totally geodesic embedded closed surface of area A in a hyperbolic three manifold is disjoint from the tori on the boundary of a rank two cusp having shortest closed geodesic of length one. In particular, if the three manifold M has n rank two cusps and k disjoint totally geodesic closed embedded surfaces, then*

$$\text{Vol}(M) > (\sqrt{3}/4)n + (4.4)k.$$

This corollary follows easily from the next

Lemma. *Suppose M is a hyperbolic three manifold containing a (rank one or two) cusp and an embedded closed totally geodesic surface Σ . Let \mathbb{T}^2 be the boundary torus or annulus having its shortest closed geodesic of length one. Then the distance*

$$d(\mathbb{T}^2, \Sigma) > \log 2.$$

A fuchsian subgroup in the group of isometries $\text{Isom}^+(\mathbb{H}^n)$ is a pair (F, X) where X is a hyperbolic hyperplane and F is a discrete subgroup keeping X and the half-spaces it bounds in \mathbb{H}^n invariant. The injectivity radius of F at $x \in X$, denoted $\text{inj}(x)$, is the largest hyperbolic disc centered at x whose F -translates are all disjoint. If x is an elliptic fixed (orbifold) point, then its injectivity radius is zero. There is a unique common orthogonal between any two disjoint (in $\overline{\mathbb{H}^n}$) hyperplanes.

The following is a hyperbolic (quantitative) version of the Klein-Maskit combination theorem for fuchsian subgroups in all dimensions. Its proof makes essential use of the combination theorem in all dimensions (see [Ma2, Ap]).

Theorem 6. *Suppose (F_1, X_1) and (F_2, X_2) are fuchsian subgroups of the full isometry group $\text{Isom}^+(\mathbb{H}^n)$ with X_1 and X_2 disjoint in \mathbb{H}^n . Let $x_1 \in X_1$ and $x_2 \in X_2$ be the endpoints of the unique common perpendicular between the hyperplanes X_1 and X_2 . If the following inequality holds*

$$(*) \quad r(\text{inj}(x_1)) + r(\text{inj}(x_2)) < d(X_1, X_2),$$

then the group $G = \langle F_1, F_2 \rangle$ is a discrete group of the second kind that, abstractly, is the free product of F_1 and F_2 . Furthermore, the hyperbolic $(n - 1)$ -hypersurfaces $\Sigma_i = X_i/F_i$ (for $i = 1, 2$) are totally geodesic boundary hypersurfaces for the hyperbolic manifold $N = \mathbb{H}^n/G$ satisfying $d_N(\Sigma_1, \Sigma_2) = d(X_1, X_2)$. The group G is torsion-free if and only if F_1 and F_2 are torsion-free.

The above theorem with very few exceptions holds when there is equality in (*). As a consequence of this, we have

Corollary 7. *Suppose Σ_1 and Σ_2 are hyperbolic $(n - 1)$ -manifolds containing embedded balls of radii R_1 and R_2 , respectively. Then there exists a hyperbolic n -manifold N having totally geodesic boundary hypersurfaces Σ_1 and Σ_2 satisfying, $d_N(\Sigma_1, \Sigma_2) = r(R_1) + r(R_2)$. \square*

The dimension three examples in Theorem 4 are constructed using the above corollary.

The proofs of the stable neighborhood theorem, its corollary, and other results on short nonsimple closed geodesics are contained in the paper [Ba3]. The proofs of the theorems on totally geodesic hypersurfaces, a more general form of Theorem 6, along with additional consequences of this approach are contained in [Ba4].

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