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A COUNTEREXAMPLE TO THE ARAKELYAN CONJECTURE

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ABSTRACT. A "self-similar" example is constructed that shows that a conjecture of N. U. Arakelyan on the order of decrease of deficiencies of an entire function of finite order is not true.

1. INTRODUCTION

Let f be an entire function and $\delta(a, f)$ denotes the Nevanlinna deficiency of f at the point $a \in \overline{C}$. Standard references are [8, 9]. (No knowedge of Nevanlinna theory is necessary to understand this paper. We really deal with a problem of potential theory.) Since $\delta(a, f) \geq 0$ and the deficiency relation of Nevanlinna states that

$$\sum_{a\in\overline{C}} \delta(a, f) \leq 2,$$

it follows that the set of deficient values, that is, $\{a : \delta(a, f) > 0\}$, is at most countable. We denote the sequence of deficiencies by $\{\delta_n\}$. In 1966 Arakelyan [2] (see also [8] or [7]) constructed the first example of an entire function of *finite order* having infinitely many deficient values. In this example the deficiencies satisfy

$$\sum_{n=1}^{\infty} \frac{1}{\log(1/\delta_n)} < \infty, \tag{1.1}$$

and he conjectured that (1.1) is true for every entire function of finite order. Another method of constructing such examples was proposed in [3], but the function in [3] also satisfies (1.1).

For meromorphic functions of finite order Weitsman [11] proved

$$\sum_{n=1}^{\infty} \delta_n^{1/3} < \infty, \tag{1.2}$$

and this is known to be best possible [9, 4]. The only known improvement of (1.2) for entire functions is due to Lewis and Wu [10]:

$$\sum_{n=1}^{\infty} \delta_n^{1/3-\epsilon_0} < \infty, \tag{1.3}$$

where ϵ_0 is an absolute constant. In fact, the value $\epsilon_0 = 2^{-264}$ is given in [10].

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In this note we will give a construction that produces an entire function of finite order having infinitely many deficiencies δ_n with the property

$$\delta_n \geq c^{-n}, \tag{1.4}$$

where $c > 1$ is a constant. Thus Arakelyan's conjecture (1.1) fails.

Of course a substantial gap still remains between the theorem of Lewis and Wu and our example. It is natural to ask whether

$$\sum_{n=1}^{\infty} \frac{1}{\log^{1+\epsilon}(1/\delta_n)} < \infty \tag{1.5}$$

is true with arbitrary $\epsilon > 0$ for entire functions of finite order.

It is more or less well known that the problem of estimating deficiencies for entire functions of finite order is equivalent to a problem of potential theory. Namely, the following statements are equivalent:

A. Given any $\rho > 1/2$ and a sequence of complex numbers a_n, there exists an entire function f of order ρ with the property $\delta(a_n, f) \geq c\delta_n$ with some constant $c > 0$.

B. There exist a bounded subharmonic function u in the annulus $A = \{z : 1 < |z| < 2\}$ and disjoint open sets $E_n \subset A$, $1 \leq n < \infty$ with the following properties:

(i) Each E_n is a union of some components of the set $\{z \in A : u(z) < 0\}$;

(ii) for every $r \in [1, 2]$

$$\int_{\{\theta : re^{i\theta} \in E_n\}} u(re^{i\theta})\, d\theta \leq -\delta_n.$$

We indicate briefly how to prove the equivalence. To prove $\mathbf{A} \to \mathbf{B}$ we take a sequence of Pólya peaks [9, p. 101] r_k for $\log M(r, f)$ and consider the sequence of subharmonic functions

$$u_k(z) = \frac{\log|f'(r_k z)|}{\log M(r_k, f)}, \qquad |z| < 2.$$

This sequence is precompact in an appropriate topology and we may take a subsequence that converges to a subharmonic function u. If f has deficient values then u satisfies (i) and (ii). See [1, 6] for details.

To prove $\mathbf{B} \to \mathbf{A}$ we apply the construction from [3] that involves an extension of u to a subharmonic function in $\mathbf{C}$ with the property of self-similarity: $u(2z) = ku(z)$, $k = \text{const} > 0$, approximation of u by the logarithm of modulus of an entire function g and performing a quasi-conformal modification on the function g that produces the entire function f satisfying **A**. It is also plausible that Arakelyan's original method could be applied directly as soon as a subharmonic function with the properties (i) and (ii) is constructed.

Remark. The above-mentioned paper of Lewis and Wu contains also the solution of a problem of Littlewood on the upper estimate of mean spherical derivative of a polynomial. The connection between the two problems seems somewhat obscure. An example that gives a lower estimate in the Littlewood's problem was constructed in [5] using some self-similar sets arising in the iteration theory of polynomials. It is interesting that the example we are going to construct now also has the property of self-similarity. Instead of iteration of a polynomial here the crucial role is played by a semigroup of Möbius transformations of the plane.

2. THE EXAMPLE

Consider the semigroup Γ generated by $z \mapsto z \pm 1$ and $z \mapsto z/2$. We have

$$\Gamma = \{\gamma_{n,k} : n = 0, 1, 2, \dots ; k = 0, \pm 1, \pm 2, \dots\},$$

where $\gamma_{n,k}(z) = 2^{-n}(z + k)$.

Denote by $S^+_{0,0}$ the square

$$S^+_{0,0} = \{z : |\Re z| \le \tfrac{3}{10}, \ |\Im z - 1| \le \tfrac{3}{10}\}$$

and set $S^+_{n,k} = \gamma_{n,k}(S^+_{0,0})$. It is easy to see that the squares $S^+_{n,k}$ are disjoint. Consider the domain

$$D_0 = \{z : 0 < \Im z < \tfrac{4}{3}\} \setminus \bigcup_{n,k} S^+_{n,k}.$$

The boundary ∂D_0 consists of the real axis, boundaries of the squares and horizontal line $l_0 = \{z : \Im z = 4/3\}$. The domain D_0 is Γ-invariant and the transformation $z \mapsto z/2$ maps D_0 onto

$$D_1 = \{z : 0 < \Im z < \tfrac{2}{3}\} \setminus \bigcup_{n,k} S^+_{n,k} \subset D_0.$$

The boundary of D_1 consists of the real axis, boundaries of some squares, and the horizontal line $l_1 = \{z : \Im z = 2/3\} \subset D_0$.

Let u be the harmonic function in D_0 that solves the Dirichlet problem

$$u(z) = 1, \qquad z \in l_0,$$

$$u(z) = 0, \qquad z \in \partial D_0 \setminus l_0.$$

This Dirichlet problem has a unique solution. So we conclude from translation invariance that

$$u(z+1) = u(z), \qquad z \in D_0. \tag{2.6}$$

It follows that the function u has a positive minimum $M^{-1} < 1$ on the line $l_1 \subset D_0$. Comparing $u(z)$ and $u(2z)$ on ∂D_1 and using the maximum principle, we conclude that $u(2z) \le Mu(z)$, $z \in D_1$, which is equivalent to

$$u(z) \le Mu(z/2), \qquad z \in D_0. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$u(\gamma_{n,k}(z)) \ge M^{-n} u(z), \qquad z \in D_0. \tag{2.8}$$

Now we are going to extend u to the strip

$$S^+ = \{z : 0 < \Im z < \tfrac{4}{3}\},$$

that is, to define u in the squares. We start by defining u in $S^+_{0,0}$. The normal derivative (in the direction of the outward normal to the boundary of the square) of u has positive infimum on $\partial S^+_{0,0}$; it tends to $+\infty$ as we approach a corner of the square. Denote by $G > 0$ the Green function for $S^+_{0,0}$ with the pole at the point $i = \sqrt{-1}$. It is clear that the normal derivative of G on the boundary of the square is bounded (it tends to zero as we approach a corner). Set

$$u(z) = -tG(z), \qquad z \in S^+_{0,0},$$

where $t > 0$. If t is small enough we obtain a subharmonic extension of u into $S^+_{0,0}$, because the jump of the normal derivative will be positive as we cross the boundary of the square from inside. Fix such t, and extend u to the remaining squares by the formula

$$u(\gamma_{n,k}(z)) = M^{-n}u(z), \qquad z \in S^+_{0,0}. \tag{2.9}$$

It follows from (2.8) that the normal derivative always has a positive jump as we cross the boundary of $S^+_{n,k}$, so the extended function is subharmonic in S^+.

Now consider the smaller squares

$$K^+_{0,0} = \{z : |\Re z| \le \tfrac{2}{7},\ |\Im z - 1| \le \tfrac{2}{7}\} \subset S^+_{0,0}, \qquad K^+_{n,k} = \gamma_{n,k}(K^+_{0,0}) \subset S^+_{n,k}.$$

It follows from (2.9) that

$$u(z) \le -\beta M^{-n}, \qquad z \in K^+_{n,k} \tag{2.10}$$

for some $\beta > 0$ and all n and k.

Now we are going to extend u to the strip $S = \overline{S^+ \cup S^-}$ where

$$S^- = \{z : -\tfrac{4}{3} < \Im z < 0\}.$$

To do this we repeat the above construction starting with the square

$$S^-_{0,0} = \{z : |\Re z - \tfrac{1}{2}| \le \tfrac{3}{10},\ |\Im z + 1| \le \tfrac{3}{10}\}$$

and using the same semigroup Γ. We obtain the squares

$$S^-_{n,k} = \gamma_{n,k}(S^-_{0,0}) \quad \text{and} \quad K^-_{n,k} = \gamma_{n,k}(K^-_{0,0}),$$

where

$$K^-_{0,0} = \{z : |\Re z - \tfrac{1}{2}| \le \tfrac{2}{7},\ |\Im z + 1| \le \tfrac{2}{7}\},$$

and the function u_1 subharmonic in S^- that satisfies the inequality similar to (2.10):

$$u_1(z) \le -\beta_1 M_1^{-n}, \qquad z \in K^-_{n,k} \tag{2.11}$$

with some $\beta_1 > 0$ and $M_1 > 1$.

Extend u to $S = \overline{S^+ \cup S^-}$ by setting $u(z) = u_1(z)$, $z \in S^-$ and $u(x) = 0$, $x \in \mathbf{R}$. The extended function u is continuous in S. We will prove that it is subharmonic in S.

Consider the strips $\Pi_n = \{z : |\Im z| < \frac{4}{3}2^{-n}\}$. Define the functions v_n in the following way: $v_n(z) = u(z)$, $z \in S \backslash \Pi_n$; v_n are continuous in S and harmonic in Π_n. Then $v_n(x) > 0$, $x \in \mathbf{R}$, and it follows from the maximum principle (applied to $\Pi_n^+ = \{z : 0 < \Im z < \frac{4}{3}2^{-n}\}$) that $v_n \geq u$ in S. We conclude that v_n are subharmonic because the sub-mean value property holds in every point of S. Furthermore, it is evident that $v_n \to u$ uniformly in S as $n \to \infty$, so u is subharmonic in S.

Denote $K_n^+ = \bigcup_k K_{n,k}^+$ and $K_n^- = \bigcup_k K_{n,k}^-$ and remark that each vertical line $\Re z = x_0$ intersects $K_n^+ \cup K_n^-$. Indeed the projection of K_n^+ onto the real axis is

$$\bigcup_{k \in \mathbf{Z}} \{x : |x - 2^{-n}k| \leq \tfrac{2}{7}2^{-n}\}$$

and the projection of K_n^- is

$$\bigcup_{k \in \mathbf{Z}} \{x : |x - 2^{-n}(k - \tfrac{1}{2})| \leq \tfrac{2}{7}2^{-n}\}.$$

It is clear that the union of these two sets is the whole real axis.

Now set $E_n = \bigcup_k (S_{n,k}^+ \cup S_{n,k}^-)$ and $K_n = K_n^+ \cup K_n^-$. Then each E_n is a union of some components of the set $\{z \in S : u(z) < 0\}$ and for every vertical line $l = \{z : \Re z = x_0\}$ the length of intersection $l \cap K_n$ is at least b^{-n} for some $b > 1$. So we have in view of (2.10) and (2.11):

$$\int_{l \cap E_n} u(x_0 + iy)\, dy \leq \int_{l \cap K_n} u(x_0 + iy)\, dy \leq -c^{-n}$$

with some constant $c > 1$.

It remains to make a change of variable $z = \frac{4}{3\varepsilon} \log \zeta$, $\zeta \in Q = \{\zeta : 1 < |\zeta| < 2,\ |\arg \zeta| < \varepsilon\}$, and to extend the function $u(z(\zeta))$ (in any desired manner) from Q to a subharmonic function in the annulus $\{1 < |\zeta| < 2\}$. The extended function will have the properties (i) and (ii) of **A** with $\delta_n \geq c^{-n}$, $c > 1$.

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