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SMOOTH STATIC SOLUTIONS OF THE EINSTEIN-YANG/MILLS EQUATION

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ABSTRACT. We consider the Einstein/Yang-Mills equations in $3+1$ space time dimensions with $\mathrm{SU}(2)$ gauge group and prove rigorously the existence of a globally defined smooth static solution. We show that the associated Einstein metric is asymptotically flat and the total mass is finite. Thus, for non-abelian gauge fields the Yang/Mills repulsive force can balance the gravitational attractive force and prevent the formation of singularities in spacetime.

1

The only static, i.e., time independent, solution to the vacuum Einstein equations for the gravitational field $R_{ij} - \frac{1}{2}Rg_{ij} = 0$ is the celebrated Schwarzschild metric that is singular at $r = 0$ [1]. Despite this defect, this solution has applicability for large r to physical problems, e.g., the perihelion shift of Mercury. Similarly, the Yang/Mills equations $d^*F = 0$, which unify electromagnetic and nuclear forces, have no static regular solutions on $\mathbb{R}^4$ [3]. Furthermore, if one couples Einstein's equations to Maxwell's equations, to unify gravity and electromagnetism

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = \sigma T_{ij}\,, \qquad d^*F = 0 \tag{1}$$

(T_{ij} is the stress-energy tensor relative to the electromagnetic field F_{ij}), the only static solution is the Reissner-Nordström metric, which is again singular at the origin [1]. Finally, the Einstein-Yang/Mills (EYM) equations, which unify gravitational and nuclear forces, were shown in [4] to have no static regular solutions in $(2+1)$ space time dimensions for any gauge group G. We announce here that the contrary holds in $(3+1)$ space-time dimensions. Indeed, with $\mathrm{SU}(2)$ gauge group (i.e., the weak nuclear force) we prove that the EYM equations (c.f. (1), where now F_{ij} is the su(2)-valued Yang/Mills field), admit

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nonsingular static solutions, whose metric is asymptotically flat, i.e., Minkowskian. (Strong numerical evidence for this conclusion was obtained by Bartnik and McKinnon [2] who also derived the relevant equations.) Thus for nonabelian gauge fields, the Yang-Mills repulsive force can balance gravitational attraction and prevent the formation of singularities in spacetime. Viewed differently from a mathematical perspective, it is the nonlinearity of the corresponding Yang/Mills equations that allows the existence of smooth solutions.

The EYM equations are obtained by minimizing the action

$$\int (-R + |F|^2)\sqrt{g}\, dx,$$

over all metrics g_{ij} having signature $(-, +, +, +)$. These equations become

$$R_{ij} = 2F_{ik}F_j^k - \tfrac{1}{2}|F|^2 g_{ij}.$$

Here R is the scalar curvature associated to the metric g_{ij} and F is the Yang-Mills curvature. These formidable equations become more tractible if we consider static symmetric solutions.

2

The problem of finding static, symmetric nonsingular solutions of the EYM equations with SU(2) gauge group can be reduced to the study of the following system of ordinary differential equations

$$r^2Aw'' + \Phi w' + w(1 - w^2) = 0, \tag{2a}$$

$$rA' + (2w^2 + 1) = 1 - \frac{(1 - w^2)^2}{r^2}, \tag{2b}$$

$$2raT' + (2w'^2A + \Phi/r)T = 0. \tag{2c}$$

Here $\Phi(r) = r(1 - A) - \frac{(1-w^2)^2}{r}$, A and T are the unknown metric coefficients, $ds^2 = -T^{-2}(r)dt^2 + A^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, and w is the "connection coefficient" relative to the sought-for connection $\alpha = w\tau_1 d\theta + [\cos\theta\tau_3 + w\sin\theta\tau_2]d\phi$, τ_1, τ_2, and τ_3 being the generators of the Lie algebra su(2). The associated curvature $F - d\alpha + \alpha\wedge\alpha$ is

$$F = w'\tau_1 dr\wedge d\theta + w'\tau_2 dr\wedge(\sin\theta d\phi) - (1 - w^2)\tau_3 d\theta\wedge(\sin\theta d\phi).$$

If $< \tau_i, \tau_j >= -2tr\tau_i\tau_j$ denotes the Killing form on su(2), and if $|F|^2 = g^{ij}g^{kl}F_{ij}F_{kl}$, then an easy calculation gives

$$|F|^2 = 2w'^2/r^2 + (1 - w^2)^2/r^4.$$

In order that our solution has finite mass, i.e., that $\lim_{r\to\infty} r(1 - A(r)) < \infty$ we require that

$$\lim_{r\to\infty}(w(r), w'(r)) \text{ be finite.} \tag{3}$$

Furthermore, asymptotic flatness of the metric means that

$$\lim_{r\to\infty}(A(r), T(r)) = (1, 1). \tag{4}$$

Finally, the conditions needed to ensure that our solution is nonsingular at $r = 0$ are

$$w(0) = 1, \qquad w'(0) = 0, \qquad A(0) = 1, \qquad T'(0) = 0.$$

One sees from (2) that the first two equations do not involve T. Thus we first solve these for A and w, subject to the above initial and asymptotic conditions.

3

We prove that under the above boundary conditions, every solution is uniquely determined by $w''(0)$; $w''(0) = -\lambda$ is a free parameter. We seek a $\lambda > 0$ such that there exists an orbit $(w(r, \lambda), w'(r, \lambda))$ that "connects two rest points." It is then not very difficult to prove that (4) will also hold.

A major difficulty is to show that the equations (2a), (2b) actually define a nonsingular orbit; i.e., that $w'(r, \lambda)$ is bounded and that $A(r, \lambda)$ remains positive. Our first result is

Theorem 1. *If* $0 \leq \lambda \leq 1$, *then in the region*

$$\Gamma = \{w^2 \leq 1, w' \leq 0\},$$

$A(r, \lambda) > 0$ *and* $w'(r, \lambda)$ *is bounded from below.*

On the other hand, we can also prove (see Figure 1)

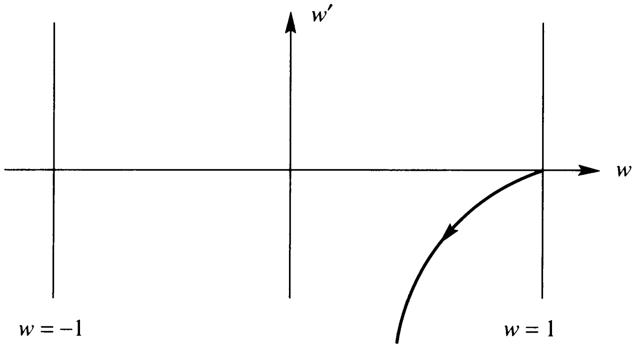

FIGURE 1

Theorem 2. *If* $\lambda > 2$, *then the solution of equations* (2a), (2b), *with initial conditions* (5) *blows up in* Γ; *i.e.,* $w'(r)$ *is unbounded.*

If λ is near zero, then by rescaling we can show that the orbit $(w(r, \lambda), w'(r, \lambda))$ exits Γ through the line $w = -1$. Furthermore, for $\lambda = 1$, numerical approximations indicate that w' becomes positive in the region $-1 < w < 0$. If this could be established rigorously, we could assert the existence of some $\bar{\lambda}$, $0 < \bar{\lambda} < 1$, for which the corresponding orbit stays in Γ for all $r \geq 0$, thereby proving (3). It would then be possible to prove that

$$\lim_{r \to \infty} (w(r, \bar{\lambda})), w'(r, \bar{\lambda})) = (-1, 0), \tag{5}$$

and as a consequence, that (4) would also hold.

4

We can give a completely rigorous proof of the existence of a connecting orbit with $\lambda < 2$, which we now outline. First Theorem 2 and the fact that for

λ near 0 the corresponding orbit exits Γ through the line $w = -1$ implies that there is a smallest $\lambda = \bar{\lambda}$ for which the orbit $(w(r, \bar{\lambda}), w'(r, \bar{\lambda}))$ does not exit Γ through this line. Thus only the following two possibilities can arise:

(P_1) There is a real number $\bar{r} > 0$ such that either (a) $w'(\bar{r}, \bar{\lambda}) = 0$, or (b) $A(\bar{r}, \bar{\lambda}) = 0$, or (c) $w'(r, \lambda)$ is unbounded for r near $\bar{r}$.

(P_2) For all $r > 0$, $w(r, \bar{\lambda}) > -1$, $w'(r, \bar{\lambda}) < 0$, and $A(r, \bar{\lambda}) > 0$.

In the case that (P_2) holds, we can show, as above, that both (6) and (7) hold. In order to rule out possibility (P_1), we consider several cases. The crucial case occurs when $A(\bar{r}, \bar{\lambda}) = 0$, $w'(r, \bar{\lambda})$ is unbounded near $r = \bar{r}$, and $\Phi(\bar{r}, \bar{\lambda}) = 0$. Now set $\overline{w} = \lim_{r \nearrow \bar{r}} w(r, \bar{\lambda})$. If $\overline{w} < 0$, then defining $v(r, \lambda) = (Aw')(r, \lambda)$, we show that v satisfies a first order ode, and we can prove that for $\lambda < \bar{\lambda}$, λ near $\bar{\lambda}$, there is an $r = r(\lambda)$ such that $v(r, \lambda) = 0$ and $w(r, \lambda) > -1$. This violates the definition of $\bar{\lambda}$. Similarly, if $\overline{w} > 0$, we can reduce this case to the previous one. Finally, the case where $\overline{w} = 0$ is dealt with by extending our solution into the complex plane and using the fact that the pair of functions $(w(r), A(r)) = (0, 1 + 1/r^2 - c/r)$ is always a solution of (2a) and (2b).

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