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A CHARACTERIZATION OF CONVEX HYPERBOLIC POLYHEDRA AND OF CONVEX POLYHEDRA INSCRIBED IN THE SPHERE

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ABSTRACT. We describe a characterization of convex polyhedra in $\mathbf{H}^3$ in terms of their dihedral angles, developed by Rivin. We also describe some geometric and combinatorial consequences of that theory. One of these consequences is a combinatorial characterization of convex polyhedra in $\mathbf{E}^3$ all of whose vertices lie on the unit sphere. That resolves a problem posed by Jakob Steiner in 1832.

In 1832, Jakob Steiner in his book [23] asked the following question:

> In which cases does a convex polyhedron have a (combinatorial) equivalent which is inscribed in, or circumscribed about, a sphere?

This was the 77th of a list of 85 open problems posed by Steiner, of which only numbers 70, 76, and 77 were still open as of last year. Apparently René Descartes was also interested in the problem (see [12]).

Several authors found families of noninscribable polyhedral types, beginning with Steinitz in 1927 (cf. [14]); all of these families later were subsumed by a theorem of Dillencourt [11]. In their 1991 book [9, problem B18], Croft, Falconer, and Guy had the following to say:

> It would of course be nice to characterize the polyhedra of inscribable type, but as this may be over-optimistic, good necessary, or sufficient, conditions would be of interest.

Here we announce a full answer to Steiner's question, in the sense that we produce a characterization of inscribable (or circumscribable) polyhedra that has a number of pleasant properties—it can be checked in polynomial time and it yields a number of combinatorial corollaries. First we note the following well-known characterization of convex polyhedra proved by Steinitz (cf. [14]).

Theorem of Steinitz. *A graph is the one-skeleton of a convex polyhedron in* $\mathbf{E}^3$ *if and only if it is a* 3*-connected planar graph.*

Note. A graph G is *k-connected* if the complement of any $k-1$ edges in G is connected.

We will call graphs satisfying the criteria of Steinitz' theorem *polyhedral graphs.*

The answer to Steiner's question stems from the following characterization of ideal convex polyhedra in hyperbolic 3-space $\mathbf{H}^3$. (See [25, 7] for the basics of hyperbolic geometry.)

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Theorem 1. *Let P be a polyhedral graph with weights $w(e)$ assigned to the edges. Let P^* be the planar dual (or Poincaré dual) of P, where the edge e^* dual to e is assigned the dual weight $w^*(e^*) = \pi - w(e)$. Then P can be realized as a convex polyhedron in $\mathbf{H}^3$ with all vertices on the sphere at infinity and with dihedral angle $w(e)$ at every edge e if and only if the following conditions hold*:

(1) $0 < w^*(e^*) < \pi$ *for all edges* e.
(2) *The sum of dual weights of edges* $e_1^*, e_2^*, \dots, e_k^*$ *bounding a face in* P^* *is equal to* 2π.
(3) *The sum of dual weights of edges* $e_1^*, e_2^*, \dots, e_k^*$ *forming a circuit that does not bound a face in* P^* *is strictly greater than* 2π.

Theorem 2. *A realization guaranteed by Theorem* 1 *is unique up to isometries of* $\mathbf{H}^3$.

Theorem 1 is proved by Rivin in [22]. It uses the methods of Aleksandrov [4] and also results and methods developed by Rivin in [18, 17] and subsequent work. A brief introduction to this theory is given in §1. A more complete treatment is given in [21].

Notes. Theorems 1 and 5 were recently extended by Rivin to general hyperbolic polyhedra of finite volume (that is, those with some finite and some ideal vertices). A characterization of ideal polyhedra with dihedral angles not greater than $\pi/2$ was given by Andreev [6]; Andreev's result is an easy consequence of Theorem 1.

The so-called projective model (or Klein model) of hyperbolic 3-space is a representation of $\mathbf{H}^3$ as the interior of the unit ball B^3 in the ordinary Euclidean 3-space $\mathbf{E}^3$. The model has the property of being *geodesic*—hyperbolic lines and planes are represented by Euclidean lines and planes, respectively. Convexity is also preserved—a convex body in $\mathbf{H}^3$ is represented by a convex body in B^3. Thus, hyperbolic convex polyhedra with all vertices on the sphere at infinity correspond precisely to convex Euclidean polyhedra inscribed in the sphere $\mathbf{S}^2 = \partial B^3$. Therefore, a polyhedron is of inscribable type exactly when it admits an edge-weighting that satisfies the condition in Theorem 1.

Furthermore (see [14]), a polyhedron is inscribable if and only if its planar dual is circumscribable, so we can sum up the characterization as follows.

Characterization R^*. A polyhedron P is of circumscribable type if and only if there exists a weighting w of its edges, such that:

(1) The weight of any edge satisfies $0 < w(e) < 1/2$.
(2) The total weight of a boundary of a face of P is equal to 1.
(3) The total weight of any circuit not bounding a face is strictly greater than 1.

Characterization R. A polyhedron P is of inscribable type if and only if its planar dual satisfies the conditions (1)–(3) of Characterization R^*.

The following theorem was proved by Smith:

Theorem 3. *Given a polyhedral graph P, we can decide whether it admits a weighting satisfying Characterization R^* in time polynomial in the number of*

vertices N. *More exactly*: *on an integer Random Access Memory* (*RAM*) *Machine* (*see* [1]) *with precision bounded by* $O(\log N)$ *bits, the running time may be bounded by* $O(N^{5.38})$ *operations.*

Skeleton of Proof. Finding the desired weighting is a linear program with the number of constraints exponential in N and the methods of [13] and [26] can be used to produce the algorithm of Theorem 3. The algorithm exploits the observation that given a graph with *prescribed* weights on the edges, it is possible to determine in polynomal time whether the weights satisfy conditions (1)–(3) of Characterization R^*. Given that, a variant of the Ellipsoid Method is seen to yield the desired algorithm. Results of [26] allow us to improve the asymptotic behavior of the algorithm somewhat; the funny looking exponent 5.38 stems from the best known complexity result for matrix inversion.

Note (added in proof). Rivin [20] recently found a much smaller (linear in N) linear program, and hence a simpler algorithm.

Hence, the two realizability questions above may also be answered in polynomial time. For some special classes of graphs, it is particularly easy to decide inscribability. We mention the following theorem of M. Dillencourt:

> Any polyhedron whose graph is 4-connected, is inscribable. Also, these graphs are circumscribable. More graph-theoretic results can be found in [10].

1. Characterization of hyperbolic polyhedra

The work of Aleksandrov [3, 4] gives a complete characterization of compact convex polyhedra in hyperbolic 3-space in terms of the intrinsic hyperbolic metric on the boundary. *Note*: Aleksandrov's work has now been extended by Rivin [19] to *ideal* convex polyhedra.

Theorem 5 gives an analogous characterization of convex hyperbolic polyhedra in terms of their dihedral angles. This also generalizes the work of Andreev [5]. A simple derivation of Andreev's results from Theorem 5 is given by Hodgson in [15].

1.1. **Compact polyhedra.** The material from this section is developed in [18]. See [21] for a more detailed exposition. Consider the *Gauss Map* G of a compact convex polyhedron P in Euclidean three-dimensional space $\mathbf{E}^3$. The map G is a *set-valued function* from P to the unit sphere $\mathbf{S}^2$, which assigns to each point p the set of outward unit normals to support planes to P at p. Thus, the whole of a face f of P is mapped under G to a single point—the outward unit normal to f. An edge e of P is mapped to a geodesic segment $G(e)$ on $\mathbf{S}^2$, whose length is easily seen to be the exterior dihedral angle at e. A vertex v of P is mapped by G to a spherical polygon $G(v)$, whose sides are the images under G of edges incident to v and whose angles are easily seen to be the angles supplementary to the planar angles of the faces incident to v; that is, $G(e_1)$ and $G(e_2)$ meet at angle $\pi - \alpha$ whenever e_1 and e_2 meet at angle α. In other words, $G(v)$ is exactly the "spherical polar" of the link of v in P. (The *link* of a vertex is the intersection of a infinitesimal sphere centered at v with P, rescaled, so that the radius is 1.)

Collecting the above observations, it is seen that $G(P)$ is combinatorially dual to P, while metrically it is the unit sphere $\mathbf{S}^2$.

Now apply a similar construction to a convex polyhedron P in $\mathbf{H}^3$. Associate to each vertex v of P a spherical polygon $G(v)$ spherically polar to the link of v in P. Glue the resulting polygons together into a closed surface, using the rule that $G(v_1)$ and $G(v_2)$ are identified isometrically whenever v_1 and v_2 share an edge.

The resulting metric space $G(P)$ is topologically $\mathbf{S}^2$ and the complex is still Poincaré dual to P. Metrically, however, it is no longer the round sphere. To see this, consider $G(f)$—the single common point of the spherical polygons $G(v_i)$, where v_i is a vertex of f. The angle of $G(v_i)$ incident to $G(f)$ is the exterior angle of f at v_i, and so by the Gauss-Bonnet Theorem, the sum of these angles is $2\pi + \text{area}(f) \neq 2\pi$. Thus $G(f)$ is a *cone-like singularity*, or a *cone point*, with *cone angle* greater than 2π. (A cone angle equal to 2π corresponds to a smooth point.)

This analogue of the Gauss map turns out to have rather remarkable properties. Here is a brief summary:

1. The image of a convex Euclidean polyhedron under the Gauss map is always the round sphere $\mathbf{S}^2$. In sharp contrast, the following theorem holds.

Theorem 4 (Compact Uniqueness). *The metric of $G(P)$ determines the hyperbolic polyhedron P uniquely (up to congruence).*

The proof of uniqueness follows the argument used by Cauchy in the proof of his celebrated rigidity theorem for convex polyhedra in $\mathbf{E}^3$ (see [8, 4, or 24]).

2. Using the hyperboloid model of hyperbolic 3-space we can construct a model of the map G, which is not unlike the well-known spherical polar map. Let $\mathbf{E}^3_1$ denote Minkowski space: $\mathbf{R}^4$ equipped with the inner product of signature $-, +, +, +$. Then $\mathbf{H}^3$ is represented by one sheet of the hyperboloid $\{x \in \mathbf{E}^3_1 \mid \langle x, x\rangle = -1\}$, which is the "sphere of radius $\sqrt{-1}$" in $\mathbf{E}^3_1$. (For a thorough discussion of the hyperboloid model of $\mathbf{H}^3$ see [25, 7].)

The *polar* P^* of a convex polyhedron $P \subset \mathbf{H}^3$ consists of all outward Minkowski unit normals to the support planes of P. Each such unit normal vector gives a point in the the *de Sitter Sphere* $\mathbf{S}^2_1 = \{x \in \mathbf{E}^3_1 \mid \langle x, x\rangle = 1\}$, which is the "sphere of radius 1" in $\mathbf{E}^3_1$. It turns out that P^* is a convex polyhedron in $\mathbf{S}^2_1$ and that the intrinsic metric of P^* is exactly $G(P)$.

Note. The de Sitter sphere $\mathbf{S}^2_1$ is a semi-Riemannian submanifold of $\mathbf{E}^3_1$ of constant sectional curvature 1. See [16] for further discussion of the geometry of $\mathbf{E}^3_1$ and semi-Riemannian manifolds in general.

3. We obtain a precise intrinsic characterization of those surfaces that can arise as $G(P)$ for a compact convex polyhedron P in $\mathbf{H}^3$. The characterization is quite easy to state:

Theorem 5. Characterization Theorem for compact polyhedra. *A metric space (M, g) homeomorphic to $\mathbf{S}^2$ can arise as the Gaussian image $G(P)$ of a compact convex polyhedron P in $\mathbf{H}^3$ if and only if the following conditions hold*:

(a) *The metric g has constant curvature 1 away from a finite collection of cone points c_i.*
(b) *The cone angles at the c_i are greater than 2π.*
(c) *The lengths of closed geodesics of (M, g) are all strictly greater than 2π.*

The necessity of (a) and (b) is immediately apparent from the above discussion of G. The necessity of (c) is based on hyperbolic version of Fenchel's theorem ("the total geodesic curvature of a hyperbolic space curve is greater than 2π") and the "polarity" model of the map G sketched in 2. See [21] for the details.

The proof of the sufficiency of conditions (a)–(c) is based on Aleksandrov's Invariance of Domain Principle (see [2, 4]), which exploits the observation that an open and closed continuous map f from a topological space A into a connected topological space B is necessarily onto.

Using this idea to prove Theorem 5 requires a careful study of the space $\mathscr{M}_n$ of metrics on $\mathbf{S}^2$ with n cone points satisfying conditions (a)–(c), of the space $\mathscr{P}_n$ of convex polyhedra in $\mathbf{H}^3$ with n faces, and of the Gauss map $G : \mathscr{P}_n \to \mathscr{M}_n$.

1.2. **Ideal polyhedra.** The theory of the previous section is extended to noncompact polyhedra in [22]. Ideal polyhedra can be viewed as "boundary points" of $\mathscr{P}_n$, and likewise Theorem 1 can be viewed as a "limiting case" of Theorem 5. In particular, a polyhedral graph P^* as in the statement of Theorem 1 can be completed to a piecewise-spherical metric on $\mathbf{S}^2$ by gluing in a standard round hemi-sphere into each face. It may be shown that this metric satisfies the conditions (a)–(c) of Theorem 5, *except* that it contains closed geodesics of length 2π, corresponding precisely to the equators of the added hemi-spheres.

Note. In [17] the *necessity* of the conditions of Theorem 1 is established without reference to the characterization of compact polyhedra.

The techniques used to prove Theorem 5 are extended to prove Theorem 1 in [22]. The proof involves geometric estimates on families of convex polyhedra in $\mathbf{H}^3$ whose vertices move away to the ideal boundary of $\mathbf{H}^3$ and beyond. The methods actually suffice to produce a characterization of polyhedra of *finite volume* in $\mathbf{H}^3$, which includes Theorem 1 and Theorem 5 as special cases. The techniques used to prove Theorem 4 give only partial uniqueness results for ideal polyhedra (see [17]). That approach also yields an algorithm for actually producing an ideal polyhedron in $\mathbf{H}^3$ with prescribed dihedral angles, which runs in time polynomial in the number of vertices of the polyhedron and the number of decimals of accuracy required. In other words this algorithm produces coordinates for a convex inscription of a graph into the unit sphere in $\mathbf{E}^3$. This is worthy of note, as the isometric embedding results of Aleksandrov et al. and Theorem 5 do *not* give an effective way to produce a polyhedron with the desired properties.

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References

1. A. V. Aho, J. E. Hopcroft, and J. D. Ullman, *The design and analysis of computer algorithms*, Addison-Wesley, Reading, MA, 1974.

2. A. D. Aleksandrov, *An application of the theorem of invariance of domain to existence proofs*, Izv. Akad. Nauk SSSR Sci. Mat. **3** (1939), 243–255. (Russian; English Summary)
3. A. D. Aleksandrov, *The intrinsic metric of a convex surface in a space of constant curvature*, Dokl. Acad. Sci. SSSR **45** (1944), 3–6.
4. A. D. Aleksandrov, *Convex polyhedra*, GITTL, Moscow, 1950. (Russian); German transl. in Akademie Verlag Berlin 1958. MR 12, 732; 19, 1192.
5. E. M. Andreev, *On convex polyhedra in Lobachevskii space*, Math. USSR Sb. **10** (1970), 413–440.
6. E. M. Andreev, *On convex polyhedra of finite volume in Lobachevskii space*, Math. USSR Sb. **12** (1970), 255–259.
7. Alan F. Beardon, *The geometry of discrete groups*, Springer-Verlag, New York, 1983.
8. A. L. Cauchy, *Sur les polygones et polyèdres*, 2nd memoir, J. École Polytech. **19** (1813), 87–98.
9. H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved problems in geometry*, Springer-Verlag, 1991.
10. M. Dillencourt and Warren D. Smith, *Graph-theoretic aspects of inscribability*, in preparation.
11. M. B. Dillencourt, *Toughness and Delaunay triangulations*, J. Discrete Comput. Geom. **5** (1990), 575–601.
12. P. J. Federico, *Descartes on polyhedra: A study of De Solidorum Elementis*, Sources in the History of Mathematics and Physical Sciences, vol. 4, Springer-Verlag, New York, 1982.
13. M. Grotschel, L. Lovasz, and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica **1** (1981), 169–197.
14. Branko Grünbaum, *Convex polytopes*, Wiley, New York, 1967.
15. C. D. Hodgson, *Deduction of Andreev's theorem from Rivin's characterization of convex hyperbolic polyhedra*, Topology 90, Proceeedings of the Research Semester in Low Dimensional Topology at O.S.U., de Gruyter Verlag (to appear).
16. Barrett O'Neill, *Semi-Riemannian geometry; with applications to relativity*, Academic Press, New York, 1983.
17. Igor Rivin, *On geometry of convex ideal polyhedra in hyperbolic 3-space*, Topology (to appear).
18. Igor Rivin, *On geometry of convex polyhedra in hyperbolic 3-space*, PhD thesis, Princeton Univ., June 1986.
19. Igor Rivin, *Intrinsic geometry of convex polyhedra in hyperbolic 3-space*, submitted.
20. Igor Rivin, *Some applications of the hyperbolic volume formula of Lobachevsky and Milnor*, submitted.
21. Igor Rivin and C. D. Hodgson, *A characterization of compact convex polyhedra in hyperbolic 3-space*, Invent. Math. (to appear).
22. Igor Rivin, *A characterization of ideal polyhedra in hyperbolic 3-space*, preprint 1992.
23. Jakob Steiner, *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*, Reimer, Berlin, 1832; Appeared in J. Steiner's Collected Works, 1881.
24. J. J. Stoker, *Geometric problems concerning polyhedra in the large*, Comm. Pure Applied Math. **21** (1968), 119–168.
25. William P. Thurston, *Geometry and topology of 3-manifolds*, Lecture notes, Princeton Univ., 1978.
26. Pravin M. Vaidya, *A new algorithm for minimizing convex functions over convex sets*, IEEE Sympos. Foundations of Computer Science, October 1989, pp. 338–343.

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