17. J. F. Voloch, *Codes and curves*, Eureka **43** (1983), 53–61.
18. M. Wirtz, *Verallegemeinerte Goppa-Codes*, Diplomarbeit, Universität Munster, 1986.

J. W. P. HIRSCHFELD
UNIVERSITY OF SUSSEX
E-mail address: mmfd4@central.sussex.ac.uk

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 2, October 1992

0273-0979/92 $1.00 + $.25 per page

Analytic pseudodifferential operators for the Heisenberg group and local solvability, by Darryl Geller. Princeton University Press, Princeton, NJ, 1990, 495 pp., $29.50. ISBN 0-691-08564-1

The subject of the book is the study of the analyticity of the solutions of partial differential equations modeled on certain invariant equations on the Heisenberg group. The importance of such equations comes from the fact that they arise naturally in complex analysis, in the study of strictly pseudoconvex boundaries or of CR structures.

An important literature already exists concerning these equations in a more classical framework, where one is interested in the existence or smoothness of solutions, and derive Sobolev or Hölder weighted inequalities. The analyticity of solutions has been less studied, although the result of P. Greiner, J. J. Kohn, and E. Stein sheds light on the problem in a striking way and makes this study unavoidable. Their result is the following: let Ω be a complex domain in $\mathbf{C}^n$ with smooth boundary X. Then the $\overline{\partial}_b$ system, i.e., the system of tangential Cauchy-Riemann differential equations on X describes the differential conditions for a function on X to be the boundary value of a holomorphic function on Ω; it can be thought of as the differential system obtained from the usual De Rham exterior differential d by restricting it to antiholomorphic vectors tangent to X: $\overline{\partial}_b$ is a quotient of the De Rham complex d.

If X bounds a strictly pseudoconvex domain, e.g., if it is the sphere bounding the unit ball of $\mathbf{C}^n$ or the equivalent paraboloïd $x_n + \overline{x}_n + \sum_1^{n-1} x_k \overline{x}_k = 0$, the equation $\overline{\partial}_b^* \omega = f$ has a global solution on X if and only if f is orthogonal to the null space of $\overline{\partial}_b$, the space of boundary values of holomorphic functions; in other words, if S is the Szegö projector, i.e., the orthogonal projector of $L^2(X)$ on the subspace $\ker \overline{\partial}_b$ of boundary values of holomorphic functions, the range of $\overline{\partial}_b^*$ is the null-space of S; this is a comparatively easy result, which just means that the range of $\overline{\partial}_b^*$ is closed and follows from J. J. Kohn's a priori estimates for the $\overline{\partial}$-Neumann problem.

When X is real analytic the theorem of Greiner, Kohn, and Stein gives a local version of this result, which is again expressed in terms of the Szegö projector S: it is known that S is globally defined by an integral kernel, but that its effect on singularities is local; in particular if X is real-analytic and strictly pseudoconvex and if f extends analytically near some point $x \in X$, then so does $S(f)$. Also the singularity of the kernel of S is a local property of the boundary. The Cauchy-Kowalewski theorem shows that the differential equation with analytic

coefficients $\overline{\partial}_b^* \omega = f$ has a solution near a point $x \in X$ if f is real analytic near x. The Greiner, Kohn, and Stein theorem states that, combined with the Cauchy-Kowalewski theorem, the Szegö projector characterization above gives exactly the right result: the equation $\overline{\partial}_b^* \omega = f$ has a solution near a point $x \in X$ if and only if $S(f)$ is real analytic in a neighborhood of x in X, with S the Szegö projector. This last property is a local property of f and clearly the result is somewhat beyond the scope of distribution or pseudodifferential methods.

It is therefore natural to study the analyticity properties of the solutions of operators such as $\overline{\partial}_b$, its adjoint $\overline{\partial}_b^*$, and the Kohn Laplacian $\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$.

Calculus on the Heisenberg group. An elementary definition of a strictly convex smooth hypersurface in $\mathbf{R}^n$ is that at any of its points it can be mapped through a linear map on a hypersurface tangent of order ≥ 3 to the sphere. A similar definition of a smooth strictly pseudoconvex boundary in $\mathbf{C}^n$ is that near any of its points, it can be mapped through a holomorphic map onto a hypersurface tangent of high order to a sphere (tangent of order ≥ 6 if $n = 2$, or 4 if $n > 2$). So one rightfully expects that the complex sphere or paraboloïd analysis will give good models for complex boundary analysis.

In this context the Heisenberg group plays and important role: it is isomorphic to the nilpotent part of the group of holomorphic transformations of the paraboloïd $x_n + \overline{x}_n + \sum_1^{n-1} x_k \overline{x}_k = 0$, which fix the origin or the point at infinity (the paraboloïd is a projective transform of the unit sphere of $\mathbf{C}^n$). The Heisenberg group acts freely on the paraboloïd, or on the sphere minus a point, so that the complex sphere or paraboloïd analysis can be viewed as part of the Heisenberg group analysis.

The Heisenberg group can be described as the set $\mathbf{H}_n = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ $(= \mathbf{R}^{2n+1})$ equipped with the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y'));$$

its infinitesimal generators are the left invariant vector fields:

$$\begin{cases} X_k = \partial/\partial x_k + 2y_k \partial/\partial t, \\ Y_k = \partial/\partial y_k - 2x_k \partial/\partial t, \\ T = \partial/\partial t, \end{cases} \tag{1}$$

which satisfy the canonical relations $[Y_j, X_k] = 4\delta_{jk} T$. The selfadjoint operators $X_k/2i$ and $Y_k/2i$ correspond to the position and momentum operators of quantum mechanics.

The Heisenberg group is a nilpotent group: its set of commutators is spanned by T and is also the center of the group. It has a fundamental one-parameter family of unitary representations in $L^2(\mathbf{R}^n)$, irreducible for $h \neq 0$, mapping the generators as follows:

$$\begin{cases} X_k \to 2iu_k, \\ Y_k \to 2h\partial/\partial u_k, \\ T \to ih, \end{cases} \tag{2}$$

where $2iu_k$ denotes the multiplication operator by $2iu_k$ on $L^2(\mathbf{R}^n)$ with coordinates $u_1, \ldots, u_n$ and ih is the multiplication operator by the constant ih.

These play an important role in the analysis of invariant differential operators, in a similar but more elaborate way as the Fourier transformation does in the analysis of differential operators with constant coefficients on $\mathbf{R}^n$.

It also has a one-parameter family of automorphisms (dilations):

$$(x, y, t) \to (\lambda x, \lambda y, \lambda^2 t), \tag{3}$$

which arises from the corresponding 2-graduation of the Lie algebra $\mathscr{G} = \mathbf{R}^{2n} \otimes \mathbf{R}$ and with respect to which the x, y are homogeneous of degree 1 and t is of degree 2. This also plays an important role in the analysis of invariant differential operators because the models are semihomogeneous, and in the study of left invariant differential or pseudodifferential operators on $\mathbf{H}_n$, one begins with those that are semihomogeneous of some degree with respect to dilations.

In connection with complex analysis one also considers the following complex combinations of the X_k, Y_k, where we have set $z_k = x_k + iy_k$:

$$\begin{cases} Z_k = \dfrac{1}{2}(X_k - iY_k) = \dfrac{\partial}{\partial z_k} + i\overline{z}_k \dfrac{\partial}{\partial t}, \\ \overline{Z}_k = \dfrac{1}{2}(X_k + iY_k) = \dfrac{\partial}{\partial \overline{z}_k} - iz_k \dfrac{\partial}{\partial t}. \end{cases} \tag{4}$$

The $\overline{Z}_k$ are the tangent Cauchy-Riemann operators when one identifies $\mathbf{H}_n$ with the paraboloïd. In quantum mechanics they are related to creation or annihilation operators.

Such semihomogeneous models provided the first examples of authentically "nonconstant coefficient" operators: among these is the well-known unsolvable H. Lewy operator, i.e., the operator $\overline{Z}_1$ above in the case $n = 1$:

$$\frac{\partial}{\partial \overline{z}} + -\frac{1}{2i} z \frac{\partial}{\partial t}, \tag{5}$$

which can be identified with the $\overline{\partial}_b$ operator of the unit sphere of $\mathbf{C}^2$.

In general we dispose of the n similar operators $\overline{Z}_k$ above. Out of their "squares" one gets the model for the Kohn Laplacian

$$\Box_b = -\sum (Z_k \overline{Z}_k + \overline{Z}_k Z_k). \tag{6}$$

Closely related to this are the operators L_α

$$L_\alpha - \frac{1}{2} \sum (Z_k \overline{Z}_k + \overline{Z}_k Z_k) - i\alpha T \qquad (\alpha \in \mathbf{C}), \tag{7}$$

which were studied by Folland and Stein. If α is of the form $\pm(n + 2k)$, k a positive integer, L_α has a nontrivial null space in L^2, whose orthogonal projector is a semihomogeneous integrodifferential operator quite similar to the Szegö projector mentioned above. Otherwise L_α is "analytic-hypoelliptic" and has a left and right elementary solution (= inverse), which is a typical example of a semihomogeneous analytic pseudodifferential operator such as those described in the book.

The book is mostly concerned with the analyticity of differential and integrodifferential operators on the Heisenberg group such as the L_α, with the construction and description of their parametrices or the projector on null space in $L^2(\mathbf{H}_n)$ when this is $\neq 0$ (i.e., α an integer of the form $\pm(n + 2k)$ in the

case of L_α) and with applications of this to operators such as $\overline{\partial}_b$, $\overline{\partial}_b^*$, $\Box_b$, and similar operators arising in CR geometry or in contact geometry.

The method consists in constructing carefully parametrices of such operators as converging perturbation series, controlling the construction sufficiently well to check the analyticity. For this the author uses thoroughly the semihomogeneity that is apparent in the formulas above, where the x's are homogeneous of degree 1 and t of degree 2. He constructs an algebra of introdifferential operators related to the algebra of pseudodifferential operators, which takes into account analyticity properties. The calculus of analytic pseudodifferential operators is a good guide but far from sufficient, since in this calculus, which is modeled on the calculus of semihomogeneous differential or convolution operators, the symbolic calculus cannot be commutative and is, therefore, more difficult. In fact this noncommutative calculus reminds me very much of the two step "hermite operator calculus," which I developed to study operators with double characteristics such as those studied here (not quoted)—although here of course the main difficulty is to control analyticity.

Contents. The book is organized as follows: the first section is a study of semihomogeneous distributions on $\mathbf{R}^n$ and of the growth conditions on the successive derivatives of the Fourier transform ensuring the analyticity. This study is interesting and the result is very nice, although it is slightly beside the main point of the book since semihomogeneous analytic-hypoelliptic operators with constant coefficients on $\mathbf{R}^n$ do not exist unless they are homogeneous in the usual sense, and the semihomogeneity needed for the book is with respect to a Heisenberg group action rather than an action of the translation group of $\mathbf{R}^n$. However, it leads to the definition of the spaces $Z_{q,j}^q$, which are much used later on: these are spaces of entire functions of finite order $(O(\exp C|z|^q))$, which have an asymptotic expansion in semihomogeneous functions defined in suitable parabolic neighborhoods of $\mathbf{R}^n$.

The next three sections are a study of operators on $\mathbf{R}^n$ invariant under the action of a suitable dilation group and homogeneous differential or integrodifferential operators on the Heisenberg group.

Chapter 6 contains the study of what the reviewer called "polynomial pseudodifferential operators" on $\mathbf{R}^n$, i.e., pseudodifferential operators $P(x, D)$ on $\mathbf{R}^n$ whose symbols $p(x, \xi)$ are asymptotic sums of functions globally homogeneous in x and ξ. The study of convolution operators on H naturally leads to the study of families of such operators. It also contains the beginning of the study of analyticity.

Chapter 7 is technically important and explains the calculus. As mentioned the "usual pseudodifferential" calculus does not work and it is replaced by the study of the Schwartz-kernel (called the core in the book). This must have an asymptotic (converging near the diagonal) expansion

$$K \sim \sum K_j(x, xy^{-1}))$$

where each $K_j(x, z)$ is analytic and "log semihomogeneous" in z and depends analytically on x. It is shown how to manipulate such operators; their calculus is more complicated (and not as explicit) as the pseudodifferential calculus and multiplication is replaced by convolution on the noncommutative group H; however, it is enough to construct parametrix in the next chapter.

This is finally applied to similar operators on strictly pseudoconvex boundaries or contact manifolds to show that when analytic operators are hypoelliptic because they are modeled on homogeneous Heisenberg group operators, they are analytic-hypoelliptic.

The main result of the last chapter is that if the boundaries X, X' of complex domains coincide and are strictly pseudoconvex and analytic near a point x, then the corresponding Szegö or Bergman kernel differ by a function that extends analytically near x. This is then applied to equations similar to the equation $\overline{\partial}_b^* f = g$, and the analogue of the result of Greiner Kohn and Stein is shown.

Conclusions. This monograph is an important work and a reasonably complete and detailed study on the subject of analyticity of solutions of partial differential equations on the Heisenberg group or on pseudoconvex boundaries; it is successfully elementary, or at least easily accessible to graduate students. The price for this is its length (500 pp.) and an inevitable and often rather heavily detailed technicality, although the technicality of the "core" is adequately compensated for by the long introduction (65 pp.), which describes very well the subject and the contents of the book. The bibliography may be found slightly short in comparison with the size of the book, but it is mostly adequate and up to date. The choice of notation is sometimes surprising or slightly misleading; for example, in this book, which is primarily concerned with differential operators, $\mathbf{D}^r$ does not denote a derivative but a semihomogeneous dilation—Rhom denotes a space of semihomogeneous functions and has nothing to do with homological algebra as any nonspecialist might think at first sight. There are several other such notations, which could be misleading to people having a mathematical culture elsewhere, but of course this is really more amusing than serious since on a purely formal point of view notation is never fundamentally significant, and any inconvenience from such notation is anyway largely compensated for by some graphically extraordinary formulas, such as formula 5.8, p. 225 or 8.34, p. 391 in which one can see long undulating sea serpents or several other things with some imagination. I do regret, however, that the work of the Japanese school, in particular that of M. Sato, T. Kawai, and M. Kashiwara on analytic partial differential equations and microanalysis, is essentially ignored: even if the cohomological notations are occasionally heavy and disagreeable to many PDE specialists, it remains deeply and fundamentally at the heart of the matter.

L. Boutet de Monvel
Universite de Paris VI
E-mail address: lbm@ccr.jussieu.fr