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Quasi-symmetric designs, by Mohan S. Shrikhande and Sharad S. Sane. London Math. Soc. Lecture Note Ser., vol. 164, Cambridge University Press, Cambridge, 1991, 225 pp, $29.95. ISBN 0-521-41407-5

There are two different visions of the theory of designs. The first comes from statistics, where a design is a way of capturing, with a small set of blocks, the regularity properties of a much larger ensemble. The second comes from finite geometry and extremal set theory. Here we might be given simple constraints on block intersection sizes with the aim of maximizing the size of the design and exploring the symmetries of extremal configurations.

A $2-(\nu, k, \lambda)$ design is a collection $\mathfrak{P}$ of subsets of a ν-set such that every member of $\mathfrak{P}$ contains k points and every pair of points is in λ blocks. If b denotes the number of blocks, and if r denotes the number of blocks containing a given point, then the identities

$$bk = \nu r \quad \text{and} \quad r(k-1) = (\nu - 1)\lambda$$

restrict the possible parameter sets. These identities are trivial in that they are obtained by elementary counting arguments.

The fundamental question in design theory is: Given ν, k, λ, does there exist a $2-(\nu, k, \lambda)$ design? It is natural to impose the restriction $k < \nu$, and in this case we have Fisher's inequality $b \geq \nu$. Designs with $b = \nu$ are called symmetric designs. In a symmetric design there is just one intersection number; two distinct blocks always intersect in λ points. Conversely it is easily shown that a 2-design with one intersection number is a symmetric design.

A design with precisely two intersection numbers is called quasi-symmetric. This is the subject of the monograph, and it is a rather special topic in the theory of designs. In scope it is similar to two other monographs in this same series; "Parallelisms of Complete Designs" by P. J. Cameron [2] and "Symmetric Designs; An Algebraic Approach" by E. S. Lander [4]. All three monographs are directed at specialists in combinatorics who want to explore a particular topic in the theory of designs. The monographs by Cameron and Lander both have a very definite point of view, and that gives them coherence. Shrikhande and Sane aim to bring out the interaction between quasi-symmetric designs, strongly regular graphs, rank 3 permutation groups, and linear codes. They have used these connections in their own research to classify quasi-symmetric designs with some additional structure, and these results are presented in the later chapters.

Fundamental to the study of quasi-symmetric designs is the observation that the block graph is strongly regular. Recall that a strongly regular graph is regular and that the number of vertices joined to two given vertices z_1 and z_2 ($z_1 \neq z_2$) only depends on whether or not z_1 and z_2 are joined. This observation appeared for the first time in a paper by S. S. Shrikhande and Bhagwandas in the statistics literature. One strength of this monograph is that it recognizes contributions to design theory by Indian mathematicians who published in statistics journals. The interplay between graphs and designs is the theme of the early chapters of this monograph. The authors describe restrictions on the design that are forced by extra structure in the block graph. I was pleased to see a chapter on the use of the Cauchy interlacing inequalities to study strongly regular graphs with strongly regular decompositions.

The most remarkable quasi-symmetric designs that are known are those associated with the Mathieu groups and the binary Golay codes. These Witt designs are described in Chapter 6. Here there is extra regularity in that the 2-design is actually a 3-design or 4-design. Any author who describes these objects has the choice of starting with the projective plane of order 4 or the [24, 12, 8] Golay code. Shrikhande and Sane start with the projective plane and, in extending that object, we see a number of computational miracles, all obtained through elementary arguments. But the miracles happen because of the Golay code and its automorphism group M_{24}. My preference would have been to start with the biggest most symmetric object at hand and to obtain the other results by specialization.

The Steiner system $S(4, 7, 23)$ is the only example of a 4-design with two intersection numbers, but Hadamard matrices provide infinitely many examples of quasi-symmetric 3-designs. In Chapter 9, Shirkhande and Sane derive the equation

$$xy(\nu-1)(\nu-2) - k(k-1)(x+y-1) + k(k-1)^2(k-2) = 0$$

that must be satisfied by the parameters of a quasi-symmetric 3-design (here x and y are the two block intersection numbers). It appears hard to find solutions to this equation that also satisfy some design-theoretic side conditions. Shrikhande and Sane then describe how the equation can be used to classify particular types of quasi-symmetric 3-designs.

The Bruck-Ryser-Chowla theorem provides a nontrivial restriction on the parameter sets of symmetric designs. Here "nontrivial" means an algebraic condition that is not a consequence of simple counting arguments. The force in this theorem is provided by the Minkowski conditions for the existence of totally isotropic spaces with respect to some nondegenerate scalar product. The focus of the book by Lander is the construction of these subspaces from the integer lattice spanned by the blocks of the design. The final chapter of the monograph under review concerns the extension of the Bruck-Ryser-Chowla theorem to nonsymmetric designs where the block intersection numbers are congruent modulo some prime. It includes all the most recent results except for those of Blokhuis and this reviewer [1] and Skinner [5].

My overall impression is that the monograph succeeds in demonstrating that the subject of quasi-symmetric designs is less narrow than one might think. I could imagine a more coherent narrative, but researchers in design theory should find this monograph to be a valuable resource. However, if I were asked to recommend a book for undergraduates, to illuminate interaction between different branches of this part of combinatorics, I would still pick "Designs, Graphs, Codes, and their Links" by Cameron and van Lint [3].

Note Added in Proof. B. Bagchi has also obtained results very similar to those appearing in [1] (B. Bagchi, *On quasi-symmetric designs*, Designs, Codes, and Cryptography, **2** (1992), 69–79).

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