# BOOK REVIEWS

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*Finiteness theorems for limit cycles*, by Yu. S. Il'yashenko. Translations of Math. Monographs, American Mathematical Society, Providence, RI, 1991, 288 + ix pp., $196.00. ISBN 0-8218-4553-5

I can still remember clearly an afternoon spent in the mathematics library in the basement of Van Vleck hall as a graduate student reading Andrew Coppel's survey of quadratic systems [3]. It was the first research paper I could easily understand, and I spent the afternoon excitedly digesting the results. There I saw for the first time the statement of Hilbert's 16th problem, and there I learned some of the remarkable properties of quadratic polynomial vector fields. Since then, mostly as an observer but sometimes as a participant, I have witnessed some of the mathematical exploration of the delightful terrain surrounding the great mountain represented by the Hilbert problem. This mountain remains unconquered, but with the publication of the book under review, a high neighboring peak has been scaled. I invite the reader on a tour of the foothills in this landscape where there are promising trailheads with distant views of these mountains.

We are going to consider analytic vector fields in the real plane, or equivalently, a system of differential equations of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y),$$

where both $P$ and $Q$ are analytic functions. In the special case that both $P$ and $Q$ are polynomials we say the vector field and the system of differential equations are "polynomial". Such systems arise in mechanics and were studied during the last century by Poincaré. Among the many great ideas he pursued, perhaps one of the most fundamental was to abandon the quest for methods of representing solutions (the subject of the usual first course in differential equations) and instead to study the qualitative features of the collection of all possible solutions of the system. This point of view leads naturally to the consideration of the geometric and topological features of the phase portrait. The analysis begins with the problem of determining the local phase portrait at each stationary point of the system and opens the vast subjects of linearization and normalization. This problem was addressed in Poincaré's dissertation where he proved several fundamental results. For example, if both eigenvalues of the linearization are in the same half plane bounded by the imaginary axis, he proved there is a local analytic change of coordinates transforming the original vector field to its linear part. The general situation is much more complicated, but it is possible, as we will indicate later, to transform locally the vector field into a normal form from which the local phase portrait can be determined. To proceed further special nonstationary trajectories called separatrices are considered. The foremost example being the stable and unstable solutions tangent to the eigenspaces of the linearization at a saddle point—a stationary point whose linearization has one positive and one negative eigenvalue. As the name suggests, these solutions separate the phase plane into invariant subsets whose phase portrait is determined by finding the destination of each separatrix and of each solution with initial value in the invariant set. Here another important discovery was made by Poincaré. The limit set of a solution need not be a stationary point, rather it can be a periodic solution called a *limit cycle* or a union of stationary points and separatrices called a *separatrix cycle*. Poincaré's *cycle limite* in its original meaning refers to both of the later possibilities. Here we emphasize that a limit cycle in modern usage is a periodic solution that is the limit set for some nonperiodic trajectory in the phase plane. We also note the only possible limit sets are those just mentioned. This is the content of the Poincaré-Bendixson theorem. Several questions arise immediately from these discoveries: How many limit sets exist? How are they situated in the phase plane? If we are interested in *polynomial* vector fields, we soon ask if the degrees of the defining polynomials are linked to the number of limit sets. This question, which surely was originally posed by Poincaré, is formulated by Hilbert as part of the 16th problem of his famous list as follows [10]:

> ... I wish to bring forward a question which, it seems to me, may be attacked by the ... method of continuous variations of coefficients, and whose answer is of ... value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (*cycles limites*) for a differential equation of the first order and degree of the form
>
> $$\frac{dy}{dx} = \frac{Y}{X},$$
>
> here $X$ and $Y$ are rational integral functions of the $n$th degree in $x$ and $y$.

This problem is soon refined. The number and possible configuration of the separatrix cycles is going to be a difficult problem, but their number can be bounded by local analysis at the stationary points. The hard problem is to count the limit cycles. Thus, our modern translation of Hilbert's problem is perhaps expressed in more familiar language: For each integer $n \geq 2$ find a uniform bound for the number of limit cycles in the phase portrait of a polynomial vector field whose defining polynomials both have degree at most $n$. But, this is presumptuous from the start. Does a polynomial vector field have a finite number of limit cycles? Poincaré himself proved the answer is yes under two assumptions: the linearization at each stationary point has eigenvalues with nonzero real part (such stationary points are called *hyperbolic*) and there are no separatrix cycles. The proof is easily constructed using two more of Poincaré's ideas. If there are infinitely many limit cycles, then they must accumulate somewhere in the finite or infinite plane. By considering the phase plane as the tangent plane at the north pole of the unit sphere, the central (Poincaré) projection onto the upper hemisphere provides a compactification of the plane after adding the equator of the sphere. The original *polynomial* vector field can be extended in a natural way to a polynomial vector field on this compactification. If there are no separatrix cycles for the extended vector field, the local behavior at the hyperbolic stationary points implies the limit cycles must accumulate on a periodic phase trajectory $\Gamma$. Next comes a very great idea of Poincaré. Cut this periodic phase trajectory with a transverse analytic curve, i.e., an analytic Poincaré section $\Sigma$. For points near an intersection of $\Gamma$ and $\Sigma$, there is a well-defined analytic function giving the point on $\Sigma$ of first return—the Poincaré map. Under our assumption, the analytic Poincaré map has an infinite number of fixed points, which have an accumulation point at the fixed point corresponding to $\Gamma$. Thus, the Poincaré map is the identity. Equivalently, $\Gamma$ is the boundary of an annulus of periodic solutions none of which are limit cycles. This contradiction proves the result. The proof in outline is trivial today, but the proof uses the now standard results about the local phase portrait near a hyperbolic stationary point and the analyticity of the solutions of differential equations with respect to initial conditions, fundamental results developed by Poincaré [15]. Also, notice that the polynomial nature of the vector field is essential to obtain the extension to the Poincaré compactification. For example, it is easy to exhibit an analytic vector field with an infinite number of limit cycles accumulating at infinity. However, the nonaccumulation argument using the return map clearly holds in any compact subset of the plane for any analytic vector field.

The finiteness problem for the general case where the vector field has nonhyperbolic stationary points or separatrix cycles was taken up by one of Poincaré's students, H. Dulac. In 1923 he published a proof of *the finiteness theorem* for analytic vector fields [5]: *An analytic vector field in any compact subset of the plane has at most a finite number of limit cycles. In particular, a polynomial vector field has at most a finite number of limit cycles.* This work was known to be difficult and its methods were really not very well understood until much later. In broad outline, Dulac follows Poincaré's reasoning and reduces the problem of nonaccumulation of limit cycles on a separatrix cycle to the study of fixed points for an associated return map. The return map is still easily defined. But, generally it is only defined on one side of a transversal section to a nonstationary point of the separatrix cycle. The argument is again reduced to proving the following proposition: If the return map has infinitely many fixed points, then it must be the identity. But, the return map defined on a separatrix cycle is generally not analytic. This is due, of course, to the existence of the stationary points on the separatrix cycle. Dulac's program to overcome this difficulty is straightforward. He defines a class of functions, which he asserts to be large enough to include the return maps for separatrix cycles and which has the property that a member of the class having a countable number of fixed points (within a suitable interval) is the identity function. Dulac's method provides a proof of the finiteness theorem; however, the details of his argument are notoriously difficult to follow.

All of this was well known to me during the time I was a graduate student. My next encounter with Dulac's problem came shortly after graduation. I met Doug Shafer at the "big" Midwest Dynamical Systems meeting in Evanston in 1979. He was visiting me during the next year and we were both attending a second meeting of the same conference series in the Spring of 1981 in Cincinnati. There I was introduced to Mauricio Peixoto. It was a real pleasure to meet someone whose theorems I had actually studied in graduate school; I was eager to engage him in conversation. I do not recall exactly how we turned to Dulac's problem, but I do remember suggesting it would be a very nice expository project for someone to translate Dulac's paper into a more readable form. Peixoto became quite animated at that point! Only later did I find out more of the details. Freddy Dumortier discovered a gap while reading Dulac's paper for a seminar at IMPA in Rio. Robert Moussu later circulated a letter asking if others accepted Dulac's proof and, in reply to Moussu, Il′yashenko mentioned that he had already noticed the same gap. Of course, Peixoto knew all about this and he quickly set me straight. I paraphrase: No! This is not work for an expositor of Dulac. Rather the proof of Dulac's theorem would be (and I quote) "mathematics of the highest order". He went on to mention that some of the best analysts in *France* agreed that the problem was open. At that moment I learned it was a very good idea to attend conferences. Soon afterwards Doug Shafer and I set to work trying to understand the situation for quadratic polynomial vector fields. We reduced the problem for quadratics in a compact subset of the finite plane to consideration of just a few types of separatrix cycles and, using special properties of quadratics, we proved limit cycles did not accumulate on all the possible separatrix cycles except for the case of a separatrix cycle consisting of a single hyperbolic saddle point (with eigenvalues $\pm\lambda$, $\lambda > 0$) and a separatrix loop. This case is not easily settled by using special properties of quadratic vector fields; we were forced to confront the main issue (albeit for the simplest case). By then Jorge Sotomayor told us Dulac's proof was correct for the case of a saddle loop and after some months of work we translated Dulac's proof into an understandable form. Using the result, we proved a quadratic vector field has at most a finite number of limit cycles in a bounded region of the plane [2]. But, there were much greater forces at work. Sotomayor, who along with R. Paterlini had already proved the space of quadratic vector fields contains an algebraic variety such that the quadratic vector fields corresponding to points in its complement have only a finite number of limit cycles [18], later showed me Il′yashenko's counterexample, which clearly defined the gap in Dulac's argument, as well as Il′yashenko's first theorem [11] on Dulac's problem: *Limit cycles do not accumulate on a separatrix cycle of an analytic vector field provided the stationary points on the separatrix cycle are all hyperbolic saddles. In particular, a polynomial vector field with only hyperbolic stationary points has a finite number of limit cycles.* Using this result, Sotomayor's student R. Bamón completed the finiteness theorem for quadratic vector fields by showing limit cycles cannot accumulate on any separatrix cycle for the extension of a quadratic vector field to the Poincaré compactification [1]. All of this was good work, but Il′yashenko's extension of Dulac's result for the saddle loop was especially striking because his proof did not use any special properties of specific classes of polynomial vector fields. Rather, his theorem went to the heart of the matter by studying the return map on a separatrix cycle for an analytic vector field. At that time Il′yashenko was already very well known. He was, I believe, a student of E. Landis who together with I. Petrovskii in the late 1950s had published a complete solution to Hilbert's problem; they gave a precise bound for the number of limit cycles of a polynomial vector field of degree $n$. The proof was not correct. In fact, Il'yashenko found counterexamples to some of the lemmas in their paper. Thus, he had already been thinking about Hilbert's problem for some time. In this regard I cannot resist mentioning that in the Foreword to the book under review, Il'yashenko notes that Dulac did not continue to publish after his 1923 memoir and speculates that perhaps Dulac found the error in his own proof and then spent the rest of his life (he died in 1955) trying to correct it. Perhaps Il'yashenko is motivated in part by similar forces. At any rate, he was inspired by the work of his mentors to consider the idea of complexification, which became a central idea of his own analysis.

After the result in [11] it was clear that Il'yashenko was close to a proof of the finiteness theorem. Also, there were mathematicians at work in France inspired by the deep work of J. Écalle. Constantly changing rumors circulated until formal announcements came from both countries: Écalle, J. Martinet, R. Moussu, and J.-P. Ramis in France [8] and Il'yashenko in Russia [13] had the proof. In the Fall of 1989 a meeting was held in the beautiful conference center at Luminy in France where both Il'yashenko and Écalle presented their results. The lectures at the conference were impressive. New ideas were presented, which will likely have a lasting impact in mathematics far beyond their application to Dulac's problem; but, no matter what the implications for the future, a summit had been reached by two men taking different paths—the finiteness theorem was proved.

The method of Écalle, including an exposition of his important ideas on resurgent functions and accelerated summation, is published in the proceedings of the Luminy conference [9]. The method of Il'yashenko is the subject of the book under review. The purpose of the book is to present a complete proof of the finiteness theorem for polynomial vector fields. Thus, of necessity, most of the book is written for the expert. However, the first chapter provides a clear outline of the main steps of his argument, as well as references to the literature and historical comments. This chapter of the book and the author's previous papers [12, 14] are highly recommended as a mathematical introduction to the subject. The contents of the book are perhaps best described by reviewing the main lines of his proof rather than the individual chapters. For this, recall that the proof can be reduced to showing that limit cycles of an analytic vector field cannot accumulate on a separatrix cycle. The first step is the resolution of the singularities on the separatrix cycle. This is a procedure for removing each degenerate stationary point and then gluing into the puncture a manifold and an extension of the original vector field on this manifold having less degenerate stationary points. Properly interpreted, this is just a change to polar coordinates and then division by appropriate powers of the radial component. The procedure is called polar blow up; the method is akin to the desingularization of an algebraic variety [6]. After desingularization, it suffices to consider separatrix cycles containing only two types of stationary points: hyperbolic saddles and stationary points whose linearization has exactly one zero eigenvalue. It is also interesting to note that the same reduction applies to the case where the separatrix cycle degenerates to a single stationary point. In particular, a consequence of the finiteness theorem is the folklore theorem that an analytic vector field does not contain any stationary points of the centerfocus type, i.e., such that each neighborhood of the stationary point contains a limit cycle surrounding the stationary point. At any rate, after desingularization, a return map is defined on a one-sided section to a nonstationary point on the separatrix cycle. This return map is representable as a composition of a finite number of section maps along the separatrix cycle. There are two types, the section maps between sections along the separatrices away from the stationary points and those between sections on the boundaries of the hyperbolic sectors at the stationary points. The first type is analytic up to the boundary while the second is generally singular at the boundary. The singular section maps are studied by normalization. For this, a change of coordinates is made at each stationary point to obtain an appropriate (Dulac) normal form. In each case the resulting normal form is integrable in the sense that there is a local first integral. These, in turn, are used to compute the asymptotics of the transition map at each hyperbolic sector. When there is only one hyperbolic sector on the separatrix cycle, the local analysis of its normal form is sufficient to settle the finiteness problem as in [2], but the analysis of compositions of the local section maps for more complicated separatrix cycles is one of the most difficult aspects of the general problem. In both cases this analysis begins by assigning asymptotic series to the normalized (singular) section maps, for example, Dulac series. A *Dulac series* is a formal series of the form

$$cx^{\nu_0} + \sum_{j=1}^{\infty} x^{\nu_j} P_j(\ln x),$$

where $c > 0$, $0 < \nu_0 < \cdots \nu_j < \cdots$, $\nu_j \to \infty$, and the functions $P_j$ are polynomials. Also, we say a germ of a function $f\colon(\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ is *semiregular* if there is a Dulac series such that for each positive integer $N$ a partial sum $S$ of the series exists satisfying $f(x) - S(x) = o(x^N)$. Dulac's argument to prove the finiteness theorem can be formalized using these definitions into two propositions: The return map is semiregular. The germ of a semiregular map is either the identity or has an isolated fixed point at the origin. It follows from these propositions that either the separatrix cycle is the boundary of an annulus of periodic solutions or there are at most a finite number of limit cycles in a collar neighborhood of the separatrix cycle. Il'yashenko was the first to show the second proposition is false: the function

$$f\colon x \mapsto x + e^{-1/x} \sin \tfrac{1}{x}$$

is semiregular with trivial Dulac series (a Dulac series consisting of the single summand $x$), but the map does not have an isolated fixed point at the origin. More importantly, he also constructed an example of an analytic vector field with a separatrix cycle (containing two saddle nodes) whose return map provides a counterexample of the same type. The correct statement of Dulac's lemma is the following: *The germ of a semiregular map has either a trivial Dulac series or an isolated fixed point at the origin.* The possibility that a semiregular map not equal to the identity can have a trivial Dulac series lies at the core of Dulac's problem; developing methods to overcome this difficulty is Il'yashenko's major achievement. His counterexample shows the *power* asymptotics represented by the Dulac series are not sufficient to determine finiteness. To remedy this deficiency, Il'yashenko considers the *exponential* asymptotics of the semiregular maps with trivial Dulac series. For the case where the stationary points on the separatrix cycle are all hyperbolic saddles, he assigns to each of them a formal series called a *Dulac exponential series* of the form

$$\nu_0\zeta + c + \sum e^{\nu_j\zeta} P_j(\zeta),$$

where $\nu_0 > 0$, $c$ is a constant, the $P_j$ are polynomials, and the $\nu_j$, which are all negative, decrease to $-\infty$. The class of functions of a complex variable (which must be defined on special types of domains called quadratic standard domains) that are asymptotic to exponential Dulac series are called *almost regular*. There are then two theorems (roughly formulated here), which together prove the finiteness theorem: (i) An almost regular map is uniquely determined by its Dulac exponential series—in particular, an almost regular map with a trivial Dulac exponential series is the identity; and (ii) a return map on a separatrix cycle all of whose stationary points are hyperbolic and with a trivial Dulac series can be extended in a unique way to an almost regular map. The exponential asymptotics, extended to an almost regular map on a region of the complex plane, are exploited to obtain the uniqueness of the representation by using the following theorem of Phragmén-Lindelöf type: *A holomorphic function bounded in the right half plane and decreasing on the positive real axis more rapidly than any exponential* $\exp(-\nu\xi)$, $\nu > 0$ *must vanish.* The case where more degenerate stationary points lie on the separatrix cycle is much more complicated but uses similar ideas. The asymptotic series must be generalized further to the *superexact asymptotic series*, which are roughly Dulac exponential series with more general coefficients. This perhaps gives a glimpse of the main lines of the proof and of the contents of this important book; however, the analysis of superexact asymptotic series, the delicate matter of the appropriate definition of the extension of section maps near hyperbolic sectors of nonhyperbolic stationary points into the complex plane (as *functional cochains*), and the proof of the uniqueness result for representation by superexact asymptotic series is a major achievement, too complicated to describe here, which requires most of the book to present.

With the finiteness theorem proved, an impressive first step on the path to a solution of Hilbert's problem has been taken, but there seems to be a long road ahead. Most of the work in this direction is for quadratic vector fields. It is known that there is a quadratic vector field with at least four limit cycles [17] and much is known about specific classes of quadratic systems. For the reader interested in further exploration of the theory of quadratic systems, I recommend [16] and, for a very interesting program to solve Hilbert's problem for quadratic vector fields, I recommend [7].

## References


1. R. Bamon, *Quadratic vector fields in the plane have a finite number of limit cycles*, Inst. Hautes Études Sci. Publ. Math. **64** (1986), 111–142.
2. C. Chicone and D. Shafer, *Separatrix and limit cycles of quadratic systems and Dulac's theorem*, Trans. Amer. Math. Soc. **278** (1983), 585–612.
3. A. Coppel, *A survey of quadratic systems*, J. Differential Equations **2** (1966), 293–304.
4. S. Diliberto, *On systems of ordinary differential equations*, Contributions to the Theory of Nonlinear Oscillations I, Ann. of Math. Stud., vol. 20, Princeton Univ. Press, Princeton, NJ, 1950, pp. 1–38.
5. H. Dulac, *Sur les cycles limites*, Bull. Soc. Math. France **51** (1923), 45–188.
6. F. Dumortier, *Singularities of vector fields*, Monografias de Math., vol. 32, IMPA, Rio de Janeiro, 1978.

7. F. Dumortier, R. Roussarie, and C. Rousseau, *Hilbert's 16th problem for quadratic vector fields*, preprint, 1992.

8. J. Écalle, J. Martinet, R. Moussu, and J-P. Ramis, *Non-accumulation des cycles-limites* I, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 375–377.

9. J. Écalle, *Finitude des cycles limites it accéléro-sommation de l'application de retour*, Lecture Notes in Math., vol. 1455, Springer-Verlag, Berlin and New York, 1990, pp. 74–159.

10. D. Hilbert, *Mathematical problems*, (M. Newton, transl.), Bull. Amer. Math. Soc **8** (1902), 437–479.

11. Yu. S. Il'yashenko, *Limit cycles of polynomial vector fields with nondegenerate singular points on the real plane*, Funct. Anal. Appl. **18** (1984), 199–209.

12. ———, *"On limit cycles" and related problems of the local theory of differential equations*, Russian Math. Surveys **40** (1985), 1–49.

13. ———, *The finiteness theorem for limit cycles*, Uspekhi Mat. Nauk. **42** (1987), 223. (Russian)

14. ———, *Finiteness theorems for limit cycles*, Russian Math. Surveys **45** (1990), 129–203.

15. H. Poincaré, *Mémore sur les courbes définies par une equation différentielle*, J. Mathématiques **7** (1881), 375–422.

16. J. Reyn, *A bibliography of the qualitative theory of quadratic systems of differential equations in the plane*, TU Delft Technical Report 89-71, 1989.

17. S-L. Shi, *A concrete example of the existence of four limit cycles for plane quadratic systems*, Sci. Sinica **23** (1980), 153–158.

18. J. Sotomayor and R. Paterlini, *Quadratic vector fields with finitely many periodic orbits*, Lecture Notes in Math., vol 1007, Springer-Verlag, Berlin and New York, 1983, pp. 753–766.

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*Quantum physics, relativity and complex spacetime*, by Gerald Kaiser. North-Holland Mathematics Studies, vol. 163, North-Holland, Amsterdam, xvi + 359 pp., $85.75. ISBN 0-444-88465-3

The history of science, technology, and mathematics suggests that it is fruitful to consider how various areas of mathematics and physics are linked to each other. As examples, Newtonian Mechanics, Quantum Mechanics, and Einstein Gravitation Theory are tied to the Theory of Ordinary Differential Equations, Hilbert Space Theory, and Riemannian Geometry, respectively. The same sort of unified overview can be applied to various parts of Engineering: Computer Science is tied to Logic and the Algebraic Theory of Languages, and Control Theory to the Lie Theory of Vector Field Systems and the Theory of Stochastic Processes.

In this very interesting book, the author investigates the interrelation between various Lie and complex function theories on the mathematical side and quantum mechanical and field-theoretic theories on the physics side. Many of the situations he considers involve the theory of deformation of Lie groups. His point of view, however, is very much that of the 'down-to-earth' mathematical