REFERENCES

[A1] S. I. Al'ber, *On n-dimensional problems in the calculus of variations in the large*, Soviet Math. Dokl. **5** (1964), 700–704.

[A2] ———, *Spaces of mappings into a manifold with negative curvature*, Soviet Math. Dokl. **9** (1967), 6–9.

[B] V. Bangert, *Geodätische Linien auf Riemannschen Mannigfaltigkeiten*, Jber. DMV **87** (1985), 39–66.

[C] Ph. Ciarlet, *Mathematical elasticity*, vol. I: *Three-dimensional elasticity*, North-Holland, Amsterdam, 1988.

[CDY] K. C. Chang, W. Y. Ding, and R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces.*

[D] B. Dacorogna, *Direct methods in the calculus of variations*, Springer-Verlag, New York, 1989.

[ES] J. Eells and J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1986), 109–160.

[GG] M. Giaquinta and E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46.

[J] J. Jost, *Two-dimensional geometric variational problems*, Wiley-Interscience, Chichester, 1991.

[K] W. Klingenberg, *Lectures on closed geodesics*, Springer-Verlag, New York, 1978.

[L] L. Lemaire, *Applications harmoniques de surfaces riemanniennes*, J. Differential Geom. **13** (1978), 51–78.

[SaU] J. Sacks and K. Uhlenbeck, *On the existence of minimal immersions of 2-spheres*, Ann. of Math. (2) **113** (1981), 1–24.

[S] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), 479–495.

[SU] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **17** (1982), 307–335; Correction in J. Differential Geom. **18** (1983), 329.

[Sy] R. Schoen and S. T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Ann. of Math. (2) **110** (1979), 127–142.

[St] M. Struwe, *Plateau's problem and the calculus of variations*, Princeton Univ. Press, Princeton, NJ, 1989.

[W] H. Wente, *Large solutions to the volume constrained Plateau problem*, Arch. Rational Mech. Anal. **75** (1980), 59–77.

[Z] E. Zeidler, *Nonlinear functional analysis and its applications*, vol. IV: *Applications to mathematical physics*, Springer-Verlag, New York, 1988.

JÜRGEN JOST
INSTITUT FÜR MATHEMATIK
E-mail address: juergen.jostgruba.rz.ruhr-uni-bochum.dbp.de

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 1993

0273-0979/93 $1.00 + $.25 per page

Differential topology and quantum field theory, by Charles Nash. Academic Press, New York, 1991, 386 pp., $59.95. ISBN 0-12-514075-4.

Over the last fifteen years there has been a renaissance in the interaction between geometry and theoretical particle physics. Here "geometry" includes but is not limited to: the differential geometry of connections, now called "gauge

theory" after the physics terminology; the global analysis of elliptic operators, including the ideas surrounding the Atiyah-Singer index theorem; the global analysis of a function on a manifold—Morse theory; the geometry of Riemann surfaces and their moduli; the geometry of special varieties in higher dimensions; symplectic geometry; and the topology of manifolds in dimensions three and four. Here "theoretical particle physics" includes quantum field theory in all of its forms—including conformal field theories in two dimensions and topological field theories in two, three, and four dimensions—and string theory. Although the book under review mostly covers these geometric and physical topics, this interaction extends to many other fields of mathematics, including the representation theory of infinite-dimensional groups and algebras, quantum groups, and category theory. On the physics side it also includes ideas from condensed matter theory. My brief account in this review will omit these additional topics. As in any such account, my choice of examples, references, and names mentioned is determined by personal bias and whim and so is by no means definitive or even representative.

Historically, geometry has always benefitted from an infusion of physical ideas. This has been true from the earliest times: Egyptian geometry was created to measure the size of fields. Later the Alexandrian school developed trigonometry to make astronomical calculations and to assist in more down-to-earth matters such as the telling of time, navigation, and geography. Certainly observations of classical mechanics (apples!), as well as Kepler's laws of planetary motion, motivated Newton in his creation of the calculus. Gauss's theory of surfaces and Riemann's fundamental ideas about higher-dimensional space were in part motivated by physics, as was much of the subsequent development of Riemannian geometry. The most important influence in this century on differential geometry is Einstein's general theory of relativity. Other parts of differential geometry developed independently but in the end relate in a fundamental way to *classical* physics. Thus the geometry of connections and the nonabelian gauge theory of Yang and Mills turn out to be intimately related. While general relativity and the classical theory of fields has permeated our geometric thinking, one great advance in twentieth century physics—Quantum Theory—which stimulated 50 years of growth in functional analysis, has until the last few years had little influence in geometry (Wigner's influence on representation theory excluded).

The renaissance under discussion began in the mid 1970s with the discovery that nonabelian gauge theory can be phrased in terms of the differential geometry of connections. One of the early achievements was the application of the Atiyah-Singer index theorem to count the dimension of the space of instantons (self-dual connections) [AHS] and the application of algebraic geometry to construct solutions [ADHM]. This early period saw many applications of global analysis and topology to quantum field theory. In the 1980s a variety of mathematical techniques, including the index theorem, were used to elucidate the "anomalies" in quantum field theory [AS]. It seems safe to say that most of the interaction during this period was an application of existing mathematics to physical problems:

$$\text{geometry} \rightarrow \text{physics}.$$

There are some notable exceptions, however. For example, Witten's formula for the global anomaly [W1] led to new developments in index theory.

During this period *classical* gauge theory made itself felt in mathematics, just by virtue of giving us a great set of equations! An influential study by Atiyah and Bott of the Yang-Mills equations in two dimensions [AB] made some advances in understanding the topology of moduli spaces of stable bundles. (Very recently ideas from *quantum* physics have been used to solve some of the outstanding problems here.[1]) The self-duality equations in four dimensions led to more startling results, mainly due to Donaldson. His thesis [D1] uses these equations to show that many topological 4-manifolds are not smoothable. One immediate consequence is the existence of exotic differentiable structures on $\mathbb{R}^4$. This has blossomed into a tremendously active area of research [DK] and has led to fundamental advances in our understanding of differential topology in four dimensions.

String theory gained prominence in the mid 1980s, and this led theoretical physicists to use a wider variety of mathematics. Young physicists were no longer trained merely in traditional mathematical topics like special functions; many could now calculate the cohomology of homogeneous spaces more effectively than their mathematical counterparts! Moreover, the mathematics they used went beyond the index theorem to all sorts of questions about Riemann surfaces and infinite-dimensional algebras. Closely related to string theory is conformal field theory in two dimensions and the more special rational conformal field theories. As these developed the interaction between the mathematics and the physics became more balanced; not only did existing mathematics apply to physical problems, but now physical ideas also contributed new ideas into mathematics:

$$\text{geometry} \leftrightarrow \text{physics}.$$

Field theory often connects parts of mathematics that mathematicians had always seen as distinct. For example, the *Verlinde conjecture* computes a Riemann-Roch formula (certain intersection pairings) on the moduli space of stable bundles mentioned above in terms of the representation theory of Kac-Moody Lie algebras. (This conjecture is now proved in some cases.) Also, the diffeomorphism group of the circle was suddenly intimately connected with the moduli space of Riemann surfaces. In both examples it is the relationship of the Lagrangian and Hamiltonian points of view, which in classical mechanics proceeds through the Legendre transform, that leads to the new insights.

With the advent of topological quantum field theory [W2, W3] and topological gravity [W4] there can be no doubt as to the centrality of these quantum ideas in certain geometrical problems:

$$\text{physics} \rightarrow \text{geometry}.$$

(I will not comment about the possible physical applications of these theories.) These topological theories have much the same mathematical structure as usual quantum theories like the Standard Model, but instead of making predictions about elementary particles that can be measured in large accelerators, they make predictions about topology that can be tested against known mathematics. Many of the new low-dimensional topological invariants developed in the 1980s—Donaldson's invariants for 4-manifolds [D2], the Jones polynomial for knots [J], Casson's invariant of a 3-manifold [AM], the Floer homology of

[1]Some of the people involved are Bertram, Donaldson, Kirwan, Szenes, Thaddeus, and Witten.

a 3-manifold [F]—are all formally defined via quantum field theory. Notice that Donaldson started with the *classical* self-duality equations and ended up discovering invariants that belong to the *quantum* theory! These topological theories have not led to major new results in geometry and topology, though I am optimistic for the future. We shall see. It certainly has led to a flurry among mathematicians,[2] who have proved many of the predictions made by these theories. Although this is certainly an important step in the absorption of these quantum ideas into mathematics, it should be said that most of these mathematical proofs use techniques that do not reflect the physical origins of the ideas. I fear that we mathematicians still understand little about the fundamental notions of quantum field theory—the path integral, the relationship to canonical quantization, locality, etc.—and how they have something to say about low-dimensional geometry and topology.

The first two-thirds of Nash's book is a rather intensive compilation of some of the mathematics that has played a role in quantum field theory and string theory. By page 90 we have already breezed through basic algebraic topology, smoothing theory for manifolds, elliptic pseudodifferential operators, sheaf cohomology, characteristic classes, and K-theory. Before long we meet the index theorem (with several variations), the algebraic geometry of Riemann surfaces, infinite-dimensional groups, and Morse theory. The last third of the book covers some basics about string theory, anomalies, conformal field theory, and topological field theory. This density of material is characteristic of the subject, which draws on many mathematical disciplines. Indeed, it is one of the strengths of the book: the reader quickly gets a feel for some mathematical ideas that permeate work in this area. However, the book also suffers through its attempt to treat so many topics. I found several wrong or misleading statements. There is a lack of expository material. One wishes for more cohesion among the different topics.

I have always believed that one of the main skills we learn in graduate school is how to sit profitably through a lecture, or to read mathematics, without understanding everything. For better or for worse it is a skill we use increasingly after graduate school! Those who have mastered this technique will profit greatly by browsing in Nash's book. Different sections can be absorbed at different levels, depending on the proclivities of the particular reader, but all will learn something about the interaction between quantum field theory and geometry. Detailed study of most topics must be undertaken elsewhere; Nash's book serves as a guide for what to study. Graduate students who tackle this book may have difficulty with the breadth of material, but I still recommend it as one of the entry points into this lively field.

References

[AM] S. Akbulut and J. McCarthy, *Casson's invariant for oriented homology 3-spheres—an exposition*, Math. Notes, vol. 36, Princeton Univ. Press, Princeton, NJ, 1990.

[AB] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser A **308** (1982), 523–615.

[ADHM] M. F. Atiyah, V. Drinfeld, N. J. Hitchin, and Yu. I. Manin, *Construction of instantons*, Phys. Lett. A **65** (1978), 185–187.

[2]It is hopeless to compile a representative list. Suffice it to say that such a list covers a wide range of traditional disciplines in mathematics.

[AHS] M. F. Atiyah, N. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461.

[AS] M. F. Atiyah and I. M. Singer, *Diracs operators coupled to vector potentials*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), 2597–2600.

[D1] S. K. Donaldson, *An application of gauge theory to four dimensional topology*, J. Differential Geom. **18** (1983), 279–315.

[D2] ———, *Polynomial invariants for smooth* 4*-manifolds*, Topology **29** (1990), 257–315.

[DK] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Univ. Press, 1990.

[F] A. Floer, *An instanton invariant for* 3*-manifolds*, Comm. Math. Phys. **118** (1989), 215–240.

[J] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[W1] E. Witten, *Global gravitational anomalies*, Comm. Math. Phys. **100** (1985), 197–229.

[W2] ———, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988), 353–386.

[W3] ———, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–400.

[W4] ———, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geom. (supplement to J. Differential Geom.) (1991), 243–310.

DANIEL S. FREED
UNIVERSITY OF TEXAS AT AUSTIN
E-mail address: dafr@math.utexas.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 1993

0273-0979/93 $1.00 + $.25 per page

Around Burnside, by A. I. Kostrikin. Ergeb. Math. Grenzgeb. (3), vol. 20, Springer-Verlag, New York, Berlin, and Heidelberg, 1990, 220 pp., $82.00. ISBN 0-387-50602-0

The restricted Burnside problem, by Michael Vaughan-Lee. London Math. Soc. Monographs (N.S.), vol. 5, Oxford University Press, Oxford, 1990, 209 pp., $57.50, ISBN 0-19-853573-2

In a now-famous paper of 1902 Burnside asked some questions that have been very influential in the development of the theory of groups. Chandler and Magnus in their *The history of combinatorial group theory* [7, p. 47] go as far as to say: "A comparison of the influence of Burnside's problem on combinatorial group theory with the influence of Fermat's last theorem on the development of algebraic number theory suggests itself very strongly." Recall that a group has exponent e if the eth power of every element is the identity. The central question can then be stated: Given positive integers d and e, is every group that has exponent e and can be generated by d elements finite? As the result of work by Novikov and Adyan (see Adyan's monograph [1]) the answer is in general no. Recently there have been announcements (by S. V. Ivanov and by Lysënok) that show that (when $d \geq 2$) the answer is no, except for finitely many e.

The two books under review have as their common theme a related question usually referred to as the restricted Burnside problem: Is there among the finite